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ON VEBLEN-WEDDERBURN SYSTEMS

JAMES R. WESSON, Vanderbilt University

1. Introduction. In 1907, Veblen and Wedderburn showed that certain finite algebras could be used in order to coordinatize some finite projective planes [4], [2, p. 407]. Such an algebra was defined as a finite set of elements 0, a, b, c, $\cdots$ closed under two operations, addition $x+y$ and multiplication xy, and obeying the following:

(1) The elements form an Abelian group (with identity 0) under addition.
(2) For $a\neq 0$, the equation $xa=b$ has a unique solution x.
(3) For $a\neq 0$, the equation $ax=b$ has a unique solution x.
(4) $0a=a0=0$. (This postulate is redundant.)
(5) $(a+b)c=ac+bc$.

The postulate (5) replaces the left distributive law given by Veblen and Wedderburn; however, a new multiplication $\cdot$ defined by $x\cdot y=yx$ changes the above system into one satisfying the distributive law $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$. Therefore the two systems are anti-isomorphic [3a, p. 253].

2. Postulates. In the finite case, the system defined in Section 1 is equivalent to that defined by Hall [3a, p. 253]. The only additional postulate given by Hall is

(2.1) For $a\neq b$, the equation $xa=xb+c$ has a unique solution x,

and this condition is implied (in the finite case) by the other five. The proof of this statement follows.

First it will be shown that all nonzero elements of the additive group have the same order. Let $x_0=0$ and let $x_{i+1}=x_i+x$ for $i=0, 1, \cdots$. If a and b are two distinct nonzero elements, then $a=bc$, where $c\neq 0$, and from (5) we have $a_i=(bc)_i=b_ic$. Therefore, $a_i=0$ if and only if $b_i=0$. This leads to the following theorem [3b, p. 22].

THEOREM 2.1. *The system defined in Section* 1 *has exactly* p^n *elements, where* p *is a prime.*

LEMMA 2.1. *If* $a\neq c$, *the system of equations* $ax+y=b$, $cx+y=d$ *has a unique solution* x, y.

Proof. Let $u-v$ mean $u+(-v)$, where $-v$ is the additive inverse of v. Then the unique solution x, y of the lemma is given by $(a-c)x=b-d$, $y=b-ax=d-cx$. (Note that for any elements u and v, we have $-(uv)=(-u)v$.)

LEMMA 2.2. *If* $a\neq b$, $x_1\neq x_2$, *and* $x_1a=x_1b+c$, *then* $x_2a\neq x_2b+c$.

Proof. Suppose the contrary, and let $x_1a=x_1b+c=f$, $x_2a=x_2b+c=g$. Then the system of equations $x_1x+y=f$, $x_2x+y=g$ has two solutions, namely $x=a$, $y=0$, and $x=b$, $y=c$. This contradicts Lemma 2.1.

THEOREM 2.2. *If $a \neq b$, the equation $xa = xb + c$ has a unique solution x.*

Proof. The only nontrivial case is when $a \neq 0$ and $b \neq 0$. Let $x_1, x_2, \cdots$ be distinct elements, and let c_i $(i = 1, 2, \cdots)$ be defined by $x_i a = x_i b + c_i$. Lemma 2.2 implies the distinctness of $c_1, c_2, \cdots$. Hence, for any c, the equation $xa = xb + c$ has exactly one solution x.

It is now clear that, in the finite case, the condition (2.1) need not be given the status of a postulate. Also, both of the postulates (4) may be dropped [1, p. 17].*

3. Veblen-Wedderburn systems with units. It will be shown how any system defined by the postulates of Section 1 can be changed into a system which satisfies the original postulates and also has a unit (multiplicative identity). That a unit can be introduced has been proved by Hall [3a, pp. 253–255]. The following differs only slightly from the approach used by Hall.

THEOREM 3.1. *Let S be a set which forms a system under the postulates of the first section, and let e be any nonzero element of S. For any elements x, y, define a new multiplication $\cdot$ by*

$$x \cdot y = xa, \quad \text{where} \quad ea = y.$$

Then S forms a system satisfying the original postulates, and also $e \cdot y = y$ for all y.

Proof. Since addition is not affected, we need only to prove that postulates (2), (3), (4), and (5) are satisfied.

(4) $0 \cdot y = 0a = 0$; $x \cdot 0 = xa$, where $ea = 0$, and therefore $a = 0 = x \cdot 0$.

(2) To establish postulate (2) it is proved that $x_1 \neq x_2$ and $y \neq 0$ imply $x_1 \cdot y \neq x_2 \cdot y$. By definition

$$x_1 \cdot y = x_1 a \quad \text{where} \quad ea = y,$$
$$x_2 \cdot y = x_2 a \quad \text{where} \quad ea = y.$$

Therefore, $x_1 a \neq x_2 a$ and $x_1 \cdot y \neq x_2 \cdot y$.

(3) To establish condition (3) it is proved that $y_1 \neq y_2$ and $x \neq 0$ imply $x \cdot y_1 \neq x \cdot y_2$.

(5) By definition $(x_1 + x_2) \cdot y = (x_1 + x_2)a$, $x_1 \cdot y = x_1 a$, and $x_2 \cdot y = x_2 a$, where $ea = y$. The right distributive law for the old operations leads to the same law for the new operations.

Finally, $e \cdot y = ea$, where $ea = y$, and hence $e \cdot y = y$.

THEOREM 3.2. *Let S be a set which forms a system under the postulates of the first section, and let e be an element such that $ey = y$ for all y. For any elements x, y, define a new multiplication $\cdot$ by*

* Since $0b + ub = (0 + u)b = ub = 0 + ub$, it follows that $0b = 0$. Further $a0 = 0$. For if $a \neq 0$, and $ax = 0$ is satisfied by $x = b \neq 0$, then the equation $yb = 0$ is satisfied by $y = a \neq 0$. But this is impossible.

$$x \cdot y = ay \quad \textit{where} \quad ae = x.$$

Then S forms a system satisfying the original postulates, and also $e \cdot x = x \cdot e = x$ for all x.

Proof. The proofs that postulates (4), (2), and (3) are satisfied are similar to the corresponding proofs under Theorem 3.1. For the proof of (5) let

$$x_1 \cdot y = a_1 y \quad \text{where} \quad a_1 e = x_1,$$

$$x_2 \cdot y = a_2 y \quad \text{where} \quad a_2 e = x_2,$$

$$(x_1 + x_2) \cdot y = ay \quad \text{where} \quad ae = x_1 + x_2.$$

Then $(a_1+a_2)e = a_1e+a_2e = x_1+x_2 = ae$, and $a_1+a_2 = a$. Therefore, $(x_1+x_2)\cdot y = ay = (a_1+a_2)y = a_1y+a_2y = x_1\cdot y+x_2\cdot y$. Finally, we need to show that $e\cdot x = x\cdot e = x$. By definition $e\cdot y = ay$ where $ae = e$. Therefore $a = e$ and $e\cdot y = y$. Also, $x\cdot e = ae$ where $ae = x$, and $x\cdot e = x$.

Marshall Hall, Jr. has proved that a finite projective plane which can be "represented by a Veblen-Wedderburn number system . . . may also be represented by a Veblen-Wedderburn number system with a unit" [3a, p. 258]. The operations given in Theorems 3.1 and 3.2 may be used in order to convert any system defined by Section 1 into a system with a unit.

A *finite Veblen-Wedderburn system* is defined as a set of elements 0, e, a, b, c, $\cdots$ subject to the following postulates:

(1), (2), (3), (4), and (5) from Section 1,

(6) $$ex = xe = x \text{ for all } x.$$

Let $x_i = x+x+\cdots+x$ for i summands, $i = 1, 2, \cdots$. It is obvious that $x_i+x_j = x_{i+j}$, and it is not difficult to show that $(x_i)_j = x_{ij} = x_{ji}$. Also, $e_ie_j = (ee_j)_i = (e_j)_i = e_{ji} = e_{ij} = e_je_i$.

4. Veblen-Wedderburn systems with a prime number of elements.

THEOREM 4.1. *A Veblen-Wedderburn system with a prime number of elements is a field.*

Proof. Since the additive order of each nonzero element is a prime p, the elements may be listed 0, e, e_2, e_3, $\cdots$, e_{p-1}, where $e_i = e+e+\cdots+e$ for i summands. Multiplication is associative and commutative since $e_i(e_je_k) = e_ie_{jk}$ and $e_ie_j = e_je_i = e_{ij}$.

5. The Veblen-Wedderburn system with eight elements. It will be proved that the Veblen-Wedderburn system with eight elements is a field. Remember that $x+x = 0$ for all x, and let 0, e, a be three distinct elements. It is easily seen that $e+a$ is a fourth element; further, aa is another element. For, if $aa = e$, then $(e+a)a = a+aa = e+a$ and $a = e$, a contradiction. If $aa = e+a$, then $(e+a)a = a+aa = a+e+a = e$. Then, if another element is called b, the entire eight ele-

ments may be called $0, e, a, e+a, b, e+b, a+b, e+a+b$, and the element ba must be given by one of the following:

$$ba = e,$$
$$ba = e + b,$$
$$ba = a + b,$$
$$ba = e + a + b.$$

If $ba=e$, then $(a+b)a=aa+ba=e+a+e=a$.
If $ba=e+b$, then $(a+b)a=aa+ba=e+a+e+b=a+b$.
If $ba=a+b$, then $(e+a+b)a=e+a+b$.
If $ba=e+a+b$, then $(e+b)a=e+b$.

But since $a+b\neq e$ and $a\neq e$, each case is impossible. Hence $aa\neq e+a$. Therefore,

THEOREM 5.1. *The elements of a Veblen-Wedderburn system of order eight may be called* $0, e, a, e+a, aa, e+aa, a+aa, e+a+aa$.

Taking the eight elements as given by the preceding theorem, and remembering that $x+x=0$ for all x, one may easily construct the addition table (Abelian Group of type 1, 1, 1). There are only two possible multiplication tables, and they are given below.

	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
e	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
a	a	aa	$a+aa$	$e+a$	e	$e+a+aa$	$e+aa$
$e+a$	$e+a$	$a+aa$	$e+aa$	$e+a+aa$	aa	e	a
aa	aa	$e+a$	$e+a+aa$	$a+aa$	a	$e+aa$	e
$e+aa$	$e+aa$	e	aa	a	$e+a+aa$	$e+a$	$a+aa$
$a+aa$	$a+aa$	$e+a+aa$	e	$e+aa$	$e+a$	a	aa
$e+a+aa$	$e+a+aa$	$e+aa$	a	e	$a+aa$	aa	$e+a$

	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
e	e	a	$e+a$	aa	$e+aa$	$a+aa$	$e+a+aa$
a	a	aa	$a+aa$	$e+aa$	$e+a+aa$	e	$e+a$
$e+a$	$e+a$	$a+aa$	$e+aa$	e	a	$e+a+aa$	aa
aa	aa	$e+aa$	e	$e+a+aa$	$e+a$	a	$a+aa$
$e+aa$	$e+aa$	$e+a+aa$	a	$e+a$	$a+aa$	aa	e
$a+aa$	$a+aa$	e	$e+a+aa$	a	aa	$e+a$	$e+aa$
$e+a+aa$	$e+a+aa$	$e+a$	aa	$a+aa$	e	$e+aa$	a

Either of the following two correspondences transforms the second table above into the first.*

$$(5.1)\qquad \begin{gathered} 0 \to 0, \quad e \to e, \quad a \to e + aa, \quad e + a \to aa, \quad aa \to e + a + aa, \\ e + aa \to a + aa, \quad e + a + aa \to e + a, \quad a + aa \to a. \end{gathered}$$

$$(5.2)\qquad \begin{gathered} 0 \to 0, \quad e \to e, \quad a \to e + a + aa, \quad e + a \to a + aa, \quad aa \to e + a, \\ e + aa \to a, \quad a + aa \to aa, \quad e + a + aa \to e + aa. \end{gathered}$$

The results of this section may be summarized.

THEOREM 5.2. *The Veblen-Wedderburn system of order eight is a field.*

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* The only correspondences which transform the second table into the first are the two given above.

NEWTON'S METHOD AND MULTIPLE ROOTS

THOMAS E. MOTT, The Pennsylvania State University

Most calculus courses today cover the so-called "Newton's Method" for finding roots of equations. It is my purpose here to describe a rigorous, analytical proof of this proposition, using only results which are available in the calculus. In addition, I will generalize the method and consider the problem of multiple roots. This extension to multiple roots will then be seen to have its most fruitful application to the solution of polynomial equations. In conclusion I will give a well known theorem on rapidity of convergence.

THEOREM 1. *Let* $\phi(x)$, $\phi'(x)$, *and* $\phi''(x)$ *be continuous,* $f(x) = x - \phi(x)/\phi'(x)$, *and suppose that* $\phi'(x) \cdot \phi''(x) \neq 0$; *all in* $[a, b]$. *Suppose further that* $\phi(a) \cdot \phi(b) < 0$. *Then the equation* $\phi(x) = 0$ *has only one root in* $[a, b]$; *and if* $x_0 \in [a, b]$ *such that* $\phi(x_0) \cdot \phi''(x_0) > 0$, *the sequence* $x_0, x_1 = f(x_0), \cdots, x_{n+1} = f(x_n), \cdots$, *converges monotonically to this root* $x = p$. *On the other hand, if* $\phi(x_0) \cdot \phi''(x_0) < 0$, p *will be between* x_0 *and* x_1, *and if* $x_1 \in [a, b]$ *the sequence* $x_1, x_2, \cdots, x_n, \cdots$, *converges monotonically to* p.

Before proceeding with the proof, let us consider the geometric implications of the restrictions which have been placed on $\phi(x)$. First of all, since $\phi''(x)$ cannot change sign in (a, b), the curve $y=\phi(x)$ can have no inflection points there. Secondly, since $\phi'(x)\neq 0$ in (a, b) the curve $y=\phi(x)$ can have no horizontal tangents there. Finally, in view of the continuity of $\phi'(x)$, this last remark implies that the curve $y=\phi(x)$ has no maximum or minimum points in (a, b).

Proof of Theorem 1. First of all, since $\phi'(x)\cdot\phi''(x)\neq 0$ in $[a, b]$, it follows from Rolle's and Taylor's theorems that $\phi(x)=0$ has exactly one simple root in $[a, b]$.

Since $\phi''(x)$ is continuous and nonzero in the interval $[a, b]$ it will have constant sign there. Therefore, since $\phi(a)$ and $\phi(b)$ are of opposite sign, so also are $\phi(a)\cdot\phi''(a)$ and $\phi(b)\cdot\phi''(b)$. Furthermore, there being only one root of the equation $\phi(x)=0$ in the interval $[a, b]$, namely at $x=p$, the function $\phi(x)\cdot\phi''(x)$ holds the same sign in each of the intervals $[a, p)$ and $(p, b]$. Therefore the function $f'(x)=\{\phi(x)\phi''(x)\}/\{\phi'(x)\}^2$ has the same sign in each of the intervals $[a, p)$ and $(p, b]$ as $\phi(a)\cdot\phi''(a)$ and $\phi(b)\cdot\phi''(b)$, respectively, have. Now select the interval in which $f'(x)>0$; suppose this to be $(p, b]$, so that $\phi(b)\cdot\phi''(b)>0$.

Let $g(x)=f(x)-x$. Since $g'(p)=-1$ and $g'(x)$ is continuous in $[a, b]$, there exists $\epsilon>0$ such that $g'(x)<0$ in $[p, p+\epsilon)$. Now $g(p)=0$ hence $g(x)<0$, and from this $f(x)<x$ and consequently $\phi(x)/\phi'(x)>0$ all in $(p, p+\epsilon)$. Since $\phi(x)$ and $\phi'(x)$ are continuous and nonzero in $(p, b]$, then $\phi(x)/\phi'(x)$ has the same sign throughout $(p, b]$. Therefore we now have $\phi(x)/\phi'(x)>0$, and consequently $f(x)<x$ in $(p, b]$. But $f(p)=p$ and $f'(x)>0$ in $(p, b]$, hence $p<f(x)<x$ there.

Let the function $f(x)$ be the generating function for an iterative sequence $\{x_n\}$ so that $x_{n+1}=f(x_n)$, and let $x_0\in[a, b]$ be chosen such that $\phi(x_0)\cdot\phi''(x_0)>0$. Then $x_0\in(p, b]$ and $p<f(x_0)<x_0$, that is $p<x_1<x_0$. Thus it readily follows by induction that $p<x_{n+1}<x_n$ for $n=0, 1, \cdots$. Therefore, as a monotone decreasing sequence bounded below by p, $\{x_n\}$ is convergent to $q\geqq p$.

Now the sequences $\{x_n\}$ and $\{f(x_n)\}$ both converge to the same limit q. But $q\in(a, b)$ and $f(x)$ is continuous in $[a, b]$. Hence $q=\lim x_n=\lim f(x_n)=\lim_{x\to q} f(x)=f(q)$. Thus $x=q$ is a solution of the equation $x=f(x)$ and consequently of the equation $\phi(x)=0$, therefore $q=p$.

Suppose that $x_0\in[a, b]$ had been chosen such that $\phi(x_0)\cdot\phi''(x_0)<0$. Further suppose that $\phi(b)\cdot\phi''(b)>0$, so that $x_0\in[a, p)$. Thus $f'(x)<0$ in $[a, p)$, and since $f(p)=p$, we have $f(x)>p$ in $[a, p)$; hence our first iteration produces $x_1=f(x_0)>p$. On the other hand, if we had assumed $\phi(a)\cdot\phi''(a)>0$, we would have $x_0\in(p, b]$ and $x_1<p$. Thus, in case $x_0\in[a, b]$ is chosen such that $\phi(x_0)\cdot\phi''(x_0)<0$, then the first iteration provides $x_1=f(x_0)$ and $\phi(x_1)\cdot\phi''(x_1)>0$. Now if $x_1\in[a, b]$, the sequence $x_1, x_2, \cdots$ converges monotonically to the root p as desired.

A slight extension of our theorem is obtained as follows:

COROLLARY. *The restriction $\phi''(x)\neq 0$ in $[a, b]$ may be replaced by the weaker hypothesis that $\phi''(x)$ be either nonnegative or nonpositive in $[a, b]$.*

Proof. All of our proofs follow exactly as before with the exception that cer-

tain $<$ and $>$ signs are replaced by $\leqq$ and $\geqq$, respectively.

Up to this point we have had the restriction $\phi'(x) \neq 0$ in $[a, b]$, so that Newton's method would appear to be useful only in finding simple roots. However an important extension may be made which permits us to apply this method, under certain conditions, when multiple roots are to be located. With this purpose in mind we prove:

THEOREM 2. *Let $\phi(x)$ have continuous derivatives of orders $1, \cdots, n$ $(n \geqq 2)$ in $[a, b]$, with $\phi^{(n)}(x) \neq 0$ there. Suppose further that the equation $\phi(x) = 0$ has a root of multiplicity n in $[a, b]$. Then if $x_0 \in [a, b]$, the sequence $x_0, x_1 = f(x_0), \cdots, x_{n+1} = f(x_n), \cdots$, converges monotonically to the root $x = p$, where $f(x) = x - \phi(x)/\phi'(x)$.*

Before proceeding with the proof of this theorem we must derive certain expressions for $\phi(x)$, $\phi'(x)$, and $\phi''(x)$ which will be required there. The function $\phi(x)$ and its first n derivatives are defined and continuous in $[a, b]$. Hence, since p is a root of multiplicity n, it follows from Taylor's theorem that $\phi(p) = \phi'(p) = \cdots = \phi^{(n-1)}(p) = 0$. Thus using Taylor's theorem again, we obtain $\phi(x) = \phi^{(n)}(\zeta_0)(x-p)^n/n!$, valid in $[a, b]$, with ζ_0 between x and p. In a similar manner, we obtain $\phi'(x) = \phi^{(n)}(\zeta_1)(x-p)^{n-1}/(n-1)!$, and if $n \geqq 3$, $\phi''(x) = \phi^{(n)}(\zeta_2)(x-p)^{n-2}/(n-2)!$, both valid in $[a, b]$, and with ζ_1 and ζ_2 between x and p.

Proof of Theorem 2. First of all we wish to show that we have isolated in $[a, b]$ a single root, multiple or simple as the case may be, of each of the equations $\phi(x) = 0, \phi'(x) = 0, \cdots, \phi^{(n-1)}(x) = 0$. Consider, for instance, the equation $\phi(x) = 0$. Suppose that there exists $\xi_0 \neq p$ in $[a, b]$ such that $\phi(\xi_0) = 0$; then, according to Rolle's theorem, there exists $\xi_1 \in [a, b]$, $\xi_1 \neq p$, ξ_0, such that $\phi'(\xi_1) = 0$. We now have $\phi'(p) = \phi'(\xi_1) = 0$; and according to Rolle's theorem there exists $\xi_2 \in [a, b]$, $\xi_2 \neq p$, ξ_0, ξ_1, such that $\phi''(\xi_2) = 0$. Continuing in this manner we finally arrive at the contradictory result that there exists $\xi_n \in [a, b]$ such that $\phi^{(n)}(\xi_n) = 0$.

Let us now consider the behavior of $\phi(x) \cdot \phi''(x)$ in $[a, b]$. Since $\phi^{(n)}(x)$ is continuous and nonzero in $[a, b]$, we have either $\phi^{(n)}(x) > 0$ or $\phi^{(n)}(x) < 0$ there. Therefore, if $n \geqq 3$ we obtain $\phi(x) \cdot \phi''(x) = \phi^{(n)}(\zeta_0) \cdot \phi^{(n)}(\zeta_2)(x-p)^{2n-2}/[(n-2)!n!] > 0$; and if $n = 2$, $\phi(x) \cdot \phi''(x) = \phi''(\zeta_0) \cdot \phi''(x)(x-p)^2/2 > 0$; both inequalities being true whenever $x \in [a, b]$ and $x \neq p$. Thus we now have $\phi(x) \cdot \phi''(x) > 0$, and consequently $f'(x) > 0$ whenever $x \in [a, b]$ and $x \neq p$.

We next consider the function $f(x)$. Since $f(x) = x - \phi^{(n)}(\zeta_0)(x-p)/\phi^{(n)}(\zeta_1)n$, valid in $[a, b]$, then $\lim_{x \to p} f(x) = p$. But $f'(x) > 0$ in $(p, b]$, hence $f(x) > p$ there.

Finally consider the function $g(x) = f(x) - x$. This function and its derivative are continuous in $[a, b]$, except at $x = p$, where neither is defined. For $n \geqq 3$, we have $g'(x) = \{(n-1) \cdot \phi^{(n)}(\zeta_0) \cdot \phi^{(n)}(\zeta_2)/n[\phi^{(n)}(\zeta_1)]^2\} - 1$, valid in $[a, b]$. For $n = 2$, we have $g'(x) = \{\phi''(\zeta_0) \cdot \phi''(x)/2[\phi''(\zeta_1)]^2\} - 1$, valid in $[a, b]$. Hence, due to the continuity of $\phi^{(n)}(x)$, we obtain $\lim_{x \to p} g'(x) = -1/n$. Therefore there

exists $\epsilon > 0$ such that $g'(x) < 0$ for $x \in (p, p+\epsilon)$. But $g(x) = -\phi(x)/\phi'(x) = -\phi^{(n)}(\zeta_0)(x-p)/n\phi^{(n)}(\zeta_1)$, valid for $x \in [a, b]$; hence $\lim_{x \to p} g(x) = 0$. Thus we obtain $g(x) < 0$ and from this $f(x) < x$ and consequently $\phi(x)/\phi'(x) > 0$, all valid in $(p, p+\epsilon)$.

Now since $\phi(x)$ and $\phi'(x)$ are continuous and nonzero in $(p, b]$, then $\phi(x)/\phi'(x)$ has the same sign throughout $(p, b]$. Therefore $\phi(x)/\phi'(x) > 0$, and consequently $f(x) < x$ in $(p, b]$. But $f(p) = p$ and $f'(x) > 0$ in $(p, b]$, hence $p < f(x) < x$ there. The remainder of the proof is merely a repetition of that of Theorem 1.

This extension of Newton's method, although interesting theoretically, may be very difficult to apply to "practical" problems. The reason for this is the difficulty of isolating roots of any given multiplicity greater than one. However, in the case of polynomials, this difficulty is readily resolved.*

Returning to the sequences of Theorem 1, we first prove the

LEMMA. *Let $\phi(x)$, $\phi'(x)$, and $\phi''(x)$ be continuous in $[a, b]$. Suppose that $\phi'(x) \neq 0$ in $[a, b]$, and let $k = $ g.l.b. $|\phi'(x)|$ and $K = $ l.u.b. $|\phi''(x)|$ for $x \in [a, b]$. Further suppose that $\phi(a) \cdot \phi(b) < 0$ so that there exists exactly one simple root $x = p$ of the equation $\phi(x) = 0$ in $[a, b]$. Then $f(x) = x - \phi(x)/\phi'(x)$ and its derivative are continuous in $[a, b]$, and if $q \in [a, b]$, $|f'(x)| \leqq [k + K(b-a)] \cdot K \cdot |q-p|/k^2$ for $x \in [a, b]$ such that $|x-p| \leqq |q-p|$.*

Proof. Since $\phi'(x)$ is continuous on the closed interval $[a, b]$ there exists $\gamma \in [a, b]$ such that $|\phi'(\gamma)| = k$. According to the mean value theorem, whenever $x \in [a, b]$ we have $\phi'(x) - \phi'(\gamma) = \phi''(\xi) \cdot (x-\gamma)$ where $\xi \in [a, b]$. Therefore $|\phi'(x)| \leqq |\phi'(\gamma)| + K \cdot |x-\gamma|$ so that $|\phi'(x)| \leqq k + K(b-a)$ on $[a, b]$.

According to the mean value theorem, whenever $x \in [a, b]$ we obtain $\phi(x) - \phi(p) = \phi'(\zeta) \cdot (x-p)$ with $\zeta \in [a, b]$. Therefore for any $q \in [a, b]$ we obtain $|\phi(x)| \leqq [k + K(b-a)] \cdot |q-p|$ for all $x \in [a, b]$ such that $|x-p| \leqq |q-p|$.

Now concerning the rapidity of convergence of the iterative sequence $[x_n]$ in Theorem 1, we have

THEOREM 3. *Suppose that $\phi(x)$ in Theorem 1 is such that $K < 2k$ and that $[a, b]$ is chosen so that $(b-a) < (2-K/k)(k^2/K^2)$. Then if x_k represents the root $x = p$ with accuracy to m decimal places, the next iteration $x_{k+1} = f(x_k)$ will represent p with accuracy to at least $2m$ decimal places.*

Proof. By hypothesis $|x_k - p| < (.5) \cdot 10^{-m}$. But $(x_{k+1} - p) = f(x_k) - f(p) = f'(\xi) \cdot (x_k - p)$ where $\xi \in [a, b]$ and $|\xi - p| < |x_k - p|$. Therefore, using the lemma, $|x_{k+1} - p| < 2 \cdot |x_k - p|^2 < (.5) \cdot 10^{-2m}$ as desired.

* See J. V. Uspensky's Theory of Equations, pp. 65–67.

MAJORANTS OF POLYNOMIAL DERIVATIVES

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Introduction. In this paper we consider the relationship between a polynomial and its derivative. S. Bernstein gave the theorem [4]:

If $P(x)$ is a polynomial of degree n, then

$$|P'|/\max|P| \leqq n/[(b-x)(x-a)]^{1/2}, \qquad a \leqq x \leqq b. \tag{1}$$

He also gave the theorem [1]:

If in the interval $a \leqq x \leqq b$, $P(x)$ is a monotonically increasing polynomial of degree $(2n+1)$, then $|P|/\max P' \geqq (1/2)(b-a)/(n+1)^2$, $a \leqq x \leqq b$.

There results, as a corollary:

If in the interval $a \leqq x \leqq b$, $P(x)$ is a monotonically increasing polynomial of degree $(2n+1)$, then

$$P'/|P| \leqq 2(n+1)^2/(b-a), \qquad a \leqq x \leqq b. \tag{2}$$

A. Markoff gave the theorem [5]:

If $P(x)$ is a polynomial of degree n then

$$|P'|/\max|P| \leqq 2n^2/(b-a), \qquad a \leqq x \leqq b. \tag{3}$$

In the neighborhood of $x=a$ and $x=b$, this is better than (1).

I. Schur imposed the condition that the polynomial vanish at the end points of the interval. He obtained the theorem [9]:

If $P(x)$ is a polynomial of degree $n \geqq 2$, with $P(a)=P(b)=0$, then

$$|P'|/\max|P| \leqq [2n \cot \pi/2n]/(b-a), \qquad a \leqq x \leqq b.$$

However, for large n, this is $O(n^2)$.

P. Erdös imposed the condition that the polynomial have only real zeros. He obtained [2] a majorant of order n, namely:

Let $P(x)$ be a polynomial of degree n. If P has only real zeros, none in (a, b), then

$$|P'|/\max|P| < \frac{en}{b-a}, \qquad a \leqq x \leqq b.$$

This is the best possible result.

Bernstein too found a majorant of order n, (except at the end points of the interval); and Erdös [2] found one of order $\sqrt{n}$, namely:

Let $P(x)$ be a real polynomial of degree n, with $|P| \leqq 1$ for $-1 \leqq x \leqq 1$, having no zero in the interior of the unit circle. Given any C between 0 and 1, there will

exist a positive N depending on C, such that for $n>N$, $|P'|<(4/c^2)\sqrt{n}$, $-1+C<x<1-C$.

The results (1), (2), (3) have been extended [8] in several directions, namely: higher derivatives, fractional derivatives, two independent variables, other functions, and other point sets. For example, S. Bernstein [1] gave the following extension to the complex variable:

Let $P(z)$ be a real or complex polynomial of degree n, with $|P|\leqq 1$ for $|z|\leqq 1$, z complex. Then $|P'|\leqq n$, $|z|\leqq 1$.

P. D. Lax [6] added the condition that $P(z)$ have no zeros inside the unit circle. Then $|P'|\leqq n/2$, $|z|\leqq 1$.

N. G. de Bruijn* gave the theorems:

Let $P(z)$ and $Q(z)$ be polynomials of a complex variable z; such that when $\operatorname{Re} z\geqq 0$, $Q\neq 0$ *and* $|P|\leqq|Q|$. *Then for* $\operatorname{Re} z\geqq 0$, $|P'|\leqq|Q'|$. Also:

Let $P(z)$ and $Q(z)$ be polynomials of a complex variable z; such that the zeros of Q lie in a closed convex region, of finite or infinite extent in the plane; and such that the degree of P does not exceed the degree of Q. If $|P|\leqq|Q|$ for all points of the boundary, then also $|P'|\leqq|Q'|$ for all points of the boundary.

In the present paper we shall derive four additional majorants of polynomial derivatives. Only in the case of the first two can we show that the constants are the best possible; that is, the inequality is invalid if the constant have a smaller value.

Positive coefficients. Consider the case of positive coefficients.

THEOREM I. *If the coefficients of a polynomial $P(x)$ of degree n are real and non-negative, then $P'/P\leqq nx^{-1}$, $x>0$. (The case $P\equiv x^n$ shows that this is a best possible result.) If, moreover, the constant term is zero, then $P'/P\geqq x^{-1}$, $x>0$. (The case $P\equiv x$ shows that this is a best possible result.)*

Proof. Write $P(x)$ as $\sum_{j=0}^{n} a_j x^{n-j}$, $a_j\geqq 0$. Then

$$P'(x)=\sum_{j=0}^{n-1}(n-j)a_j x^{n-j-1}$$

$$=\frac{1}{x}\sum_{j=0}^{n-1}(n-j)a_j x^{n-j}\begin{cases}\leqq \dfrac{n}{x}\displaystyle\sum_{j=0}^{n-1}a_j x^{n-j}\leqq \dfrac{n}{x}P(x),\\[2ex] \geqq \dfrac{1}{x}\displaystyle\sum_{j=0}^{n-1}1\cdot a_j x^{n-j}=P(x)/x.\end{cases}$$

This completes the proof.

Bounded ratios of coefficients. The majorant of Theorem I can be halved, if

* Inequalities concerning polynomials in the complex domain, Nederl. Akad. Wetensch., Proc., vol. 50, 1947, pp. 1265–1272.

the conditions are tightened, as we now show:

THEOREM II. *Define* $P(x)=\sum_{j=0}^{n} c_j x^{n-j}$, $c_j>0$. *If* $0<c\leqq c_j/c_{j-1}$, $(j=1,\cdots,n)$, *then* $P'/P\leqq(1/2)nx^{-1}$, $0<x\leqq c$.

The constant is the best possible.

Proof. $xP'=\sum_{j=0}^{n-1}(n-j)c_jx^{n-j}=\sum_{j=0}^{n}(n-j)c_jx^{n-j}$. Now by Chebyshev's inequality [3], if $a_j\geqq 0$ and $b_j\geqq 0$ are monotonic sequences, one increasing and one decreasing, and if $p_j\geqq 0$, then $\sum p_ja_j\cdot\sum p_jb_j\geqq\sum p_ja_jb_j\cdot\sum p_j$.

When $p_j=1$, this becomes $\sum_{j=0}^{n}a_j\cdot\sum_{j=0}^{n}b_j\geqq(n+1)\sum_{j=0}^{n}a_jb_j$. Since $x\leqq c_j/c_{j-1}$, then $c_jx^{n-j}\geqq c_{j-1}x^{n-(j-1)}$. Thus c_jx^{n-j} *increases* as j increases. On the other hand, $n-j$ *decreases* as j increases.

Define $a_j=n-j$ and $b_j=c_jx^{n-j}$.

By Chebyshev's inequality,

$$\sum_{j=0}^{n}(n-j)\cdot\sum_{j=0}^{n}c_jx^{n-j}\geqq(n+1)\sum_{j=0}^{n}(n-j)c_jx^{n-j}.$$

Therefore, $\sum_{j=0}^{n}(n-j)\cdot\sum_{j=0}^{n}c_jx^{n-j}\geqq(n+1)xP'$. This is $(1/2)n(n+1)P\geqq(n+1)xP'$.

It remains to consider the constant $n/2$. When $P=x^n+\cdots+x+1$, at $x=1$,

$$xP'/P=\frac{n+(n-1)+\cdots+2+1}{\sum_{j=0}^{n}1}=n/2$$

Hence $n/2$ is the best constant.

Bounded coefficients. Consider now the case of bounded coefficients.

THEOREM III. *Define* $P(x)=x^n+\sum_{j=1}^{n}c_jx^{n-j}$. *If* $|c_j|\leqq c-1$, $c>1$, $(j=1,\cdots,n)$, *then* $|P'/P|\leqq n(x-(1/2)c)^2/[x(x-c)^2]$, $x>c$.

Proof. There exist constants a_j, $(j=1,\cdots,n)$, real or complex, such that $P(x)=\prod_{j=1}^{n}(x+a_j)$. Thus the zeros of $P(x)$ are $-a_i$, $(i=1,\cdots,n)$.

Suppose for some i, that $|a_i|>1$. Then for this i, we have

$$|a_i|^n\leqq\sum_{j=1}^{n}|c_j|\cdot|a_i|^{n-j}\leqq(c-1)\sum_{j=1}^{n}|a_i|^{n-j}$$

$$=(c-1)(|a_i|^n-1)/(|a_i|-1)<(c-1)|a_i|^n/(|a_i|-1).$$

Divide by $|a_i|^n$, which is >1. There results $1<(c-1)/(|a_i|-1)$, and so $|a_i|<c$.

As a consequence, either $|a_i|\leqq 1$ or $|a_i|<c$, for all i, $(i=1,\cdots,n)$. Hence in any case, $|a_i|<c$ for all i. Since $x>c$, then $x>|a_i|$, $(i=1,\cdots,n)$. Now $\log P=\sum_{j=1}^{n}\log(x+a_j)$. Differentiate.

Thus $P'/P=\sum_{j=1}^{n}(x+a_j)^{-1}$. Then

$$|P'/P| \leqq \sum_{j=1}^{n} |x + a_j|^{-1} \leqq \sum_{j=1}^{n} (x - |a_j|)^{-1}.$$

Now if $0 < \alpha \leqq \alpha_j \leqq A$ and $0 < \beta \leqq \beta_j \leqq B$, $(j = 1, \cdots, n)$, then by [7],

$$(\sum \alpha_j^2)(\sum \beta_j^2)/(\sum \alpha_j \beta_j)^2 \leqq (\alpha\beta + AB)^2/(4\alpha\beta AB).$$

Define now $\alpha = \sqrt{x-c}$, $\beta = 1/\sqrt{x}$, $A = \sqrt{x}$, $B = 1/\sqrt{x-c}$, $\alpha_j = \sqrt{x - |a_j|}$, $\beta_j = 1/\sqrt{x - |a_j|}$. There results $(1/n^2)[\sum (x - |a_j|)][\sum (x - |a_j|)^{-1}] \leqq [(x-c)+x]^2/[4(x-c)x]$. Hence $\sum (x - |a_j|)^{-1} \leqq n(x - (1/2)c)^2/[x(x-c)^2]$. The theorem follows at once.

Increasing coefficients. Theorem I tells us that when the coefficients are positive, then $xP'/P \leqq n$, $x > 0$. Theorem II tells us that if moreover the ratios of the coefficients are bounded, then $xP'/P \leqq n/2$, $0 < x \leqq c$.

We shall now find conditions such that

$$xP'/P \leqq (n+1)/k, \qquad (k \geqq e).^*$$

THEOREM IV. *Define* $P(x) = \sum_{j=0}^{n} c_j x^{n-j}$, $c_j > 0$. *If* $c_j \geqq c_{j-1}$, $(j = 1, \cdots, n)$, *then*

$$P'/P \leqq \frac{n+1}{k} x^{-1}, \qquad 0 < x \leqq \exp\left(-\frac{k}{e}\right), \qquad k \geqq e.$$

Proof. Define $F(y) \equiv (1/e) - (1/y) \log y$. Then $F' \equiv y^{-2}(\log y - 1) \geqq 0$ for $\log y \geqq 1$, that is, for $y \geqq e$. Hence $F(y)$ is increasing for $y \geqq e$.

Now $F(e) = 0$. Therefore, $F(y) \geqq 0$, $y \geqq e$. Thus $1/e \geqq (1/y) \log y$ for $y \geqq e$. Consequently $y/e \geqq \log y$, and so $\exp y/e \geqq y$, $y \geqq e$. When $j = 0, \cdots, n-1$, then $k(n-j) \geqq e$. As a result, we may take $y = k(n-j)$, $(j = 0, \cdots, n-1)$. Therefore,

$$\left(\exp \frac{k}{e}\right)^{n-j} = \exp \frac{k(n-j)}{e} \geqq k(n-j), \qquad (j = 0, \cdots, n-1).$$

Hence

$$P' \equiv x^{-1} \sum_{j=0}^{n-1} (n-j) c_j x^{n-j} \leqq \frac{1}{k} x^{-1} \sum_{j=0}^{n-1} c_j \left(x \exp \frac{k}{e}\right)^{n-j} \leqq \frac{1}{k} x^{-1} \sum_{j=0}^{n} c_j.$$

Now by Chebyshev's inequality [3], if $a_j \geqq 0$ and $b_j \geqq 0$ are monotonic sequences, both increasing, and if $p_j \geqq 0$, then

$$\sum p_j a_j \cdot \sum p_j b_j \leqq \sum p_j a_j b_j \cdot \sum p_j.$$

By hypothesis c_j is increasing, and x^{n-j} increases with j since $x < 1$. Set $a_j = c_j$, $b_j = x^{n-j}$, and $p_j = 1$.

As a result $\sum_{j=0}^{n} c_j \cdot \sum_{j=0}^{n} x^{n-j} \leqq (n+1) \sum_{j=0}^{n} c_j x^{n-j}$. Therefore,

* We originally proved the theorem for $k = 3$, and are indebted to the referee for pointing out that the same proof can be used for $k \geqq e$.

$$P \equiv \sum_{j=0}^{n} c_j x^{n-j} \geqq \frac{1}{n+1} \sum_{j=0}^{n} c_j \cdot \sum_{j=0}^{n} x^{n-j} \geqq \frac{1}{n+1} \sum_{j=0}^{n} c_j.$$

Hence

$$P'/P \leqq \frac{n+1}{k} x^{-1}.$$

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THE PARTIAL SUMMATION OF SERIES BY MATRIX METHODS

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1. Introduction. In a recently published article [1], the author discussed certain properties of a sequence of interesting matrices. Further study of these matrices led to the deduction of a number of interesting relationships involving binomial coefficients, such as the relationships (1), (2), (3), (4), *etc.* exhibited below.

While most of these relationships are not essentially new, and may be found in more general form elsewhere [2], it is the matrix technique which is used in deriving them which represents a novel approach. If this technique could be applied to establish general theorems of a similar nature, it would prove to be a useful analytical tool.

2. Some relationships for binomial coefficients. To illustrate the method let us consider the following identities:

$$(1)\qquad \sum_{i=1}^{n}(-1)^{n-i}\binom{n+i-1}{n}\binom{n}{i}=1.$$

$$(2)\qquad \sum_{i=1}^{n}(-1)^{i}(i+j-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}=0, \qquad 2\leqq j\leqq n.$$

$$(3)\qquad (-1)^{j-1}\binom{n+j-1}{n}\binom{n}{j}\sum_{i=1}^{n}\Big[(-1)^{i}(i+j-1)^{-1}\times(i+k-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}\Big]=\delta_{kj}, \qquad 2\leqq j,\ k\leqq n.$$

$$(4)\qquad \binom{n+j-1}{n}\binom{n}{j}\Big[(-1)^{n-j}+\sum_{i=1}^{n}(-1)^{i+j-1}i(i-1)(i+j-1)^{-1}\times(i+k-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}\Big]=\delta_{jk}.$$

These identities are not new. The first two, for example, follow directly from Theorems I and II applicable to polynomials

$$\phi(x)=\sum_{k=0}^{n}a_kx^k$$

of nth degree:

THEOREM I.

$$\sum_{i=0}^{r}(-1)^{i}\phi(i)\binom{r}{i}=\begin{cases}0 & \text{for } n<r\\ (-1)^{n}n!a_n & \text{for } n=r.\end{cases}$$

THEOREM II.

$$\sum_{i=0}^{n}(-1)^{i}(i+x)^{-1}\phi(y-i)\binom{n}{i}=x^{-1}\phi(x+y)\binom{n+x}{n}^{-1}.$$

Theorem I is well known* and is discussed in most books dealing with finite differences, while Theorem II is exhibited in [3]. Putting $\phi(x)=\binom{n+x-1}{n}$ in Theorem I leads to (1). Setting $y=0$, and $\phi(x)=\binom{n-x-1}{n}$ in Theorem II, results in $\sum_{i=0}^{n}(-1)^{i}(i+x)^{-1}\binom{n+i-1}{n}\binom{n}{i}=x^{-1}\binom{n-x-1}{n}\binom{n+x}{n}^{-1}$. Now letting $x=j-1$, and restricting j to the interval $2\leqq j\leqq n$ establishes relation (2). The relationships (3) and (4) follow in a similar manner from this generalization of Theorem II:

* The form in which Theorems I, II, III are stated was indicated by the referee.

THEOREM III. *If* $\phi(x)=\sum_{k=0}^{n+1} a_k x^k$, *then*

$$\sum_{i=0}^{n}(-1)^i(i+z)^{-1}(i+x)^{-1}\phi(y-i)\binom{n}{i}$$
$$=(n+1)^{-1}(x-z)^{-1}\left[\phi(z+y)\binom{n+z}{n+1}^{-1}-\phi(x+y)\binom{n+x}{n+1}^{-1}\right].$$

3. Application of the matrix method. Let us now illustrate the use of the matrix technique to establish our relationships (1) through (4).

Inspection of the identities in question seems to indicate the feasibility of considering them to be of the form

$$\sum_{i=1}^{n} a_{ji}^{(n)}\cdot\alpha_{ik}^{(n)}=\delta_{jk}, \tag{5}$$

where the $a_{ji}^{(n)}$ denote elements of an n-dimensional matrix A_n, and $\alpha_{ik}^{(n)}$ the elements of its inverse A_n^{-1}. The occurrence of the term $(i+j-1)^{-1}$ in the relationships (2), (3), (4), but not in (1), further suggests the choice of $a_{1i}^{(n)}=1$, $a_{ij}^{(n)}=(i+j-1)^{-1}$, so that

$$A_n=\begin{bmatrix}1 & 1 & \cdots & 1\\ 2^{-1} & 3^{-1} & \cdots & (n+1)^{-1}\\ 3^{-1} & 4^{-1} & \cdots & (n+1)^{-1}\\ \vdots & \vdots & & \vdots\\ n^{-1} & (n+1)^{-1} & \cdots & (2n-1)^{-1}\end{bmatrix}. \tag{6}$$

The problem of proving the relationships for the binomial coefficients thus resolves itself in the problem of inverting the A_n, and the subsequent testing of the satisfaction of the relations (1), *etc.*, interpreted in the form (5).

In turns out that* [1]

$$\alpha_{i1}=(-1)^{n-i}\binom{n+i-1}{n}\binom{n}{i}, \tag{7}$$

$$\alpha_{ij}=(-1)^{i+j-1}j(j-1)(i+j-1)^{-1}\binom{n+i-1}{n}\binom{n}{i}\binom{n+j-1}{n}\binom{n}{j} \tag{8}$$

for $j\geqq 2$.

The following alternative proof of these inversion formulas, due to D. J. Newman, is based on the relation $AA^{-1}=I$, which the α_{ij} must necessarily satisfy. Consequently,

* For convenience in writing, the indication of the fixed index n has been deleted in the following.

$$\sum_{i=1}^{n} \alpha_{ij} = \delta_{1j} \qquad j = 1, \cdots, n, \tag{9}$$

$$\sum_{i=1}^{n} (k + i - 1)^{-1}\alpha_{ij} = \delta_{kj} \qquad j = 1, \cdots, n;\ k = 2, \cdots, n. \tag{10}$$

(i) The case $j=1$. We have, by (9) and (10),

$$\sum_{i=1}^{n} \alpha_{i1} = 1, \tag{11}$$

$$\sum_{i=1}^{n} (k + i - 1)^{-1}\alpha_{i1} = 0 \quad \text{for} \quad k = 2, \cdots, n. \tag{12}$$

The α_{i1} may thus be considered to be the coefficients of the partial fraction expansion of a certain rational function:

$$f_1(x) = \sum_{i=1}^{n} \alpha_{i1}(x + i - 1)^{-1}. \tag{13}$$

By (12) this function $f_1(x)$ must vanish at $x=2, \cdots, n$; because of (11) $f_1(x)$ must asymptotically approach x^{-1} as x becomes unbounded. It may be concluded that

$$f_1(x) = (x - 2) \cdots (x - n)[x(x + 1) \cdots (x + n - 1)]^{-1}. \tag{14}$$

Consequently, it is seen that

$$\alpha_{i1} = \frac{(-i + 1 - 2)(-i + 1 - 3) \cdots (-i + 1 - n)}{(-i + 1)(-i + 2) \cdots (-i + i - 1)\cdot(-i + i + 1) \cdots (-i + n)}$$

$$= (-1)^{n-i}\binom{n + i - 1}{n}\binom{n}{i},$$

as claimed in (7).

(ii) The case $j>1$. Here, from (9) and (10),

$$\sum_{i=1}^{n} \alpha_{ij} = 0, \qquad j = 2, \cdots, n, \tag{15}$$

$$\sum_{i=1}^{n} (k + i - 1)^{-1}\alpha_{ij} = \delta_{kj}, \qquad k, j = 2, \cdots, n. \tag{16}$$

The appropriate rational functions are now

$$f_j(x) \equiv \sum_{i=1}^{n} \alpha_{ij}(x + i - 1)^{-1}.$$

Conditions (15) and (16) indicate that

$$f_j(x) = c_j(x-2)\cdots(x-j+1)(x-j-1)\cdots(x-n)\cdot[x(x+1)\cdots(x+n-1)]^{-1},$$

with c_j denoting a constant.

Putting $x=j$, and utilizing condition (16) leads to

$$c_j = j(j+1)\cdots(j+n-1)[(j-2)\cdots(j-n)]^{-1},$$

where in the brackets $[\cdots]$ the term $j-j$ is omitted.

Since the α_{ij} are the coefficients of the partial fraction expansion of $f_j(x)$, it follows that

$$\alpha_{ij} = \frac{c_j(-i+1-2)\cdots(-i+1-n)}{(-i+1)(-i+2)\cdots(-i+i-1)(-i+i+1)\cdots(-i+n)},$$

or, equivalently,

$$\alpha_{ij} = (-1)^{i+j-1}\frac{j(j-1)}{i+j-1}\binom{n+i-1}{n}\binom{n}{i}\binom{n+j-1}{n}\binom{n}{j},$$

as claimed in (8).

4. The binomial identities. Having established the expressions (7) and (8) for the α_{ij}, the identities (1) through (4) follow immediately. Inserting the formulas (7) in (11) leads to (1), and putting (7) into (12) results in (2).

Similarly, inserting the expressions (8) in (16) leads to (3).

Making use of $A^{-1}A=I$, one obtains,

$$\alpha_{i1} + \sum_{j=2}^{n} \alpha_{ij}(j+k-1)^{-1} = \delta_{ik} \tag{17}$$

for $i, k=1, \cdots, n$. The substitution of (7) and (8) into (17) results directly in (4).

The essential part of this approach thus lies clearly in the proper choice of a "generating" matrix A_n. Careful inspection of the relationships to be established may frequently provide a sufficient number of clues for the proper composition of such a generating matrix.

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ON BOUNDS FOR CHARACTERISTIC VALUES OF A PRODUCT OF MATRICES

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Let $A=(a_{ij})$ be a matrix of complex numbers; set $S(A)=\max_i \sum_j |a_{ij}|$; $T(A)=S(A^*)$. From theorems associated with the work of Hadamard, Geršgorin, Perron, it is known that for every characteristic root λ of A the relations $|\lambda| \leqq S(A)$, $|\lambda| \leqq T(A)$ must hold. Now consider the product $P=(p_{ij})$ of the matrices A, $B=(b_{ij})$. From the rule for multiplying matrices, we have $p_{ij}=\sum_k a_{ik}b_{kj}$, so that

$$\sum_j |p_{ij}| = \sum_j \Big|\sum_k a_{ik}b_{kj}\Big| \leqq \sum_j \sum_k |a_{ik}|\,|b_{kj}|$$

$$= \sum_k |a_{ik}| \Big(\sum_j |b_{kj}|\Big) \leqq S(B) \sum_k |a_{ik}| \leqq S(B)S(A),$$

i.e., $S(P) \leqq S(B)S(A)$. Thus we have proved the

THEOREM 1. *Let A, B be two square matrices of complex numbers; $P=AB$. Then every characteristic root λ of P satisfies $|\lambda| \leqq S(A)S(B)$; $|\lambda| \leqq T(A)T(B)$, and indeed for any fixed α, $0 \leqq \alpha \leqq 1$, we have $|\lambda| \leqq \{S(A)S(B)\}^{\alpha}\{T(A)T(B)\}^{1-\alpha}$.*

We put these bounds forward since they are easily computed from the entries a_{ij}, b_{ij} of the factors A, B.

By using the quantities $S_i(A)=\sum_k |a_{ik}|$, $T_i(A)=S_i(A^*)$, the above results can be further refined. Indeed Barankin showed that every characteristic root λ of AB satisfies $|\lambda| \leqq \max_i \{S_i(AB)\}^{\alpha}\{T_i(AB)\}^{1-\alpha}$, for every α, $0 \leqq \alpha \leqq 1$, and when this is combined with $S_i(P) \leqq S(B)S_i(A)$, the conclusion of Theorem 1 can be improved to read

THEOREM 2. $|\lambda| \leqq S(B)^{\alpha}T(A)^{1-\alpha} \max_i \{S_i(A)^{\alpha}T_i(B)^{1-\alpha}\}$.

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* I am indebted to the referee for his suggestions in rewording my original note and also to Professor S. M. Shah.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following problems were given the contestants in the seventeenth annual William Lowell Putnam mathematical competition:*

Part I

1. The normals to a surface all intersect a fixed straight line. Show that the surface is a portion of a surface of revolution.
2. A uniform wire is bent into a form coinciding with the portion of the curve $y=e^x$, $0 \leqq x \leqq a$, $a>1$, and the line segment $a-1 \leqq x \leqq a$, $y=e^a$. The wire is then suspended from the point $(a-1, e^a)$ and a horizontal force F is applied at the point $(0, 1)$ to hold the wire in coincidence with the curve and segment. Assuming the x-axis is horizontal, show that the force F is directed to the right.
3. A and B are real numbers and k is a positive integer. Show that

$$\left| \frac{\cos kB \cos A - \cos kA \cos B}{\cos B - \cos A} \right| < k^2 - 1$$

whenever the left side is defined.†

4. $P(z)$ is a complex polynomial whose roots (as points in the Argand plane) can be covered by a closed circular disc of radius R. Show that the roots of $nP(z)-kP'(z)$ can be covered by a closed circular disc of radius $R+|k|$, where n is the degree of $P(z)$, k is any complex number, and $P'(z)$ is the derivative of $P(z)$.
5. Given n points in the plane, show that the largest distance determined by these points cannot occur more than n times.
6. $S_1 = \ln a$, and $S_n = \sum_{i=1}^{n-1} \ln (a - S_i)$, $n > 1$.

Show that

$$\lim_{n \to \infty} S_n = a - 1.$$

7. Each member of a set of circles in the xy-plane is tangent to the x-axis and no two of the circles intersect. Show that
   (a) the points of tangency can include all the rational points on the axis but
   (b) the points of tangency cannot include all the irrational points.

* The results were annouced in this MONTHLY, vol. 64, 1957, pp. 486–488.

† The questions committee originally had a "$\leqq$" in this question, but, in some manner, the "$=$" was lost before the question reached the director. The omission was discovered in reading the proofs, but the chairman of the committee decided that it was just as well to give the contestants the opportunity to discover the error and correct it.

Part II

1. Consider the determinant $|a_{ij}|$ of order 100 with $a_{ij}=i\times j$. Prove that if the absolute value of each of the 100! terms in the expansion of this determinant is divided by 101, then the remainder in each case is 1.
2. If facilities for division are not available, it is sometimes convenient in determining the decimal expansion of $1/A$, $A>0$, to use the iteration $X_{k+1}=X_k(2-AX_k)$, $k=0, 1, 2, \cdots$, where X_0 is a selected "starting" value. Find the limitations, if any, on the starting value X_0 in order that the above iteration converge to the desired value $1/A$.
3. For $f(x)$ a positive, monotone decreasing function defined in $0\leqq x\leqq 1$, prove that

$$\frac{\int_0^1 xf^2(x)dx}{\int_0^1 xf(x)dx} \leqq \frac{\int_0^1 f^2(x)dx}{\int_0^1 f(x)dx}.$$

4. Let $a(n)$ be the number of representations of the positive integer n as the sums of 1's and 2's taking order into account. For example, since

$$4 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2 = 1+1+1+1,$$

then $a(4)=5$. Let $b(n)$ be the number of representations of n as the sum of integers greater than 1, again taking order into account and counting the summand n. For example, since $6=4+2=2+4=3+3=2+2+2$, we have $b(6)=5$. Show that for each n, $a(n)=b(n+2)$.
5. With each subset X of a set is associated a second subset $f(X)$. The association is such that whenever X contains Y then $f(X)$ contains $f(Y)$. Show that for some set A, $f(A)=A$.
6. The curve $y=f(x)$ passes through the origin with a slope of 1. It satisfies the differential equation $(x^2+9)y''+(x^2+4)y=0$. Show that it crosses the x-axis between

$$x=\frac{3}{2}\pi \quad \text{and} \quad x=\sqrt{\frac{63}{53}}\pi.$$

7. Let C be a closed convex planar disc bounded by a regular polygon. Show that for each positive integer n there exists a set of points $S(n)$ in the plane such that each n points of $S(n)$ can be covered by C, but $S(n)$ itself cannot be covered by C.

*Solutions of the Problems.** The following solutions are not taken from any of the contestants' papers, but generally embody ideas used by many contestants. In the case of Part I, No. 3 and Part II, No. 7, no contestant had a completely correct solution. The solutions of these two problems were furnished by members of the questions committee. The presentation here is intended as a brief sketch of the method of proof rather than a model of a detailed proof such as is expected from the contestants.

Part I

1. Let L be the line intersected by all the normals to the surface, and consider a plane section of the surface by a plane perpendicular to L at O. If P is any point on the intersection of the plane and the surface and if P is not on L, then the normal to the surface projects into the line OP. It follows that OP is normal to the curve in which the plane cuts the surface. An easy discussion shows that if the normals to a plane curve all pass through a fixed point, then the curve is a circle with the fixed point as center. The conclusion follows.

2. Let F_1 be the x-component of the horizontal force which just suffices to hold the wire in the required position. The force acting at the point $(a-1, e^a)$ will be just sufficient to make the sum of all the forces acting on the wire equal to zero. The sum of the moments about this point must also vanish. This requires $F_1(e^a-1)=g\left[\int_0^a (x-a+1)\sqrt{1+e^{2x}}dx+1/2\right]$, where g is the gravitational constant. Thus it is sufficient to show that the right member of this equation is positive for $a>1$. Let $f(a)$ denote the bracketed quantity above. It follows that $f(0)>0$. Then $f'(a)=\sqrt{1+e^{2a}}-\int_0^a \sqrt{1+e^{2x}}dx$ and $f''(a)=-1/\sqrt{1+e^{2a}}>-e^{-a}$. Thus $f'(a)=f'(0)+\int_0^a f''(x)dx \geqq \sqrt{2}+\int_0^a (-e^{-x})dx>0$. Thus f is increasing for $a>0$ and the result follows.

3. We first prove that $|\sin rx| \leqq r|\sin x|$, where r is a positive integer. To prove this, note that $|\sin rx| = |\sin\{(r-1)x+x\}| = |\sin(r-1)x\cos x + \cos(r-1)x \sin x| \leqq |\sin(r-1)x| + |\sin x|$, whence the required relation follows by induction. Then

$$\left|\frac{\sin rx \sin sy}{\sin x \sin y}\right| \leqq rs,$$

or more symmetrically,

$$\frac{1}{2}\left|\frac{\sin rx \sin sy + \sin sx \sin ry}{\sin x \sin y}\right| \leqq rs.$$

Let $r=k+1, s=k-1, x=(A+B)/2, y=(A-B)/2$, and the above inequality becomes (after a few elementary trigonometric identities) the required inequality with "$<$" replaced by "$\leqq$".

* These solutions are published solely for the information of interested persons. Neither the editor, nor the director of the competition, nor the paper grader will enter into any correspondence concerning them.

4. If $P(z)=A\prod_{i=1}^{n}(z-r_i)$ is the factored form of the polynomial, then $n-kP'(z)/P(z)=n-k\sum_{i=1}^{n}1/(z-r_i)$. If $|z|>R+|k|$, it follows at once that the foregoing expression is not zero and the conclusion is immediate.

5. The result is clearly true for $n=2$ and $n=3$. Assume $n>3$ and that n points determine $n+1$ segments of maximum length L. Some point P of the given points must serve as end point of three or more of these segments, for if not there could be at most n such segments. The circle with center P and radius L has on it at least three of the given points and all the given points on the circle must lie on an arc which subtends a central angle not exceeding 60°. Let Q and R be those points of the given set which are on this arc and are farthest apart. All other points of the given set must belong to the intersection of the circular discs with centers at P, Q, and R and common radius L. An easy discussion shows that S, which is on the arc QR, can not be at distance L from any of the given points except P. Thus if S is deleted we have $n-1$ points determining n largest distances, and the induction follows.

6. On the range $x\leqq a-1$ the function $x+\ln(a-x)$ is increasing and its values lie between x and $a-1$. Since for any $a>0$, $\ln a\leqq a-1$, as is easily shown, it follows that the sequence $\{S_n\}$ is monotone increasing and bounded above by $a-1$, and hence convergent. The relation $S_{n+1}=S_n+\ln(a-S_n)$ implies, by continuity, that the limit is $a-1$.

7. Let the rationals be enumerated in a sequence r_n, $n=1, 2, \cdots$. Take a circle tangent to the x-axis at the point $(r_1, 0)$ with radius 1. If nonintersecting circles have been constructed with the ith circle tangent to the x-axis at $(r_i, 0)$, $i=1, \cdots, n$, then since the point $(r_{n+1}, 0)$ is outside all these circles, it is at positive distance from their union. A circle tangent to the x-axis at the $(n+1)$th point and not intersecting any of the previous circles is then readily constructed, and the induction is complete.

If C is a set of nonintersecting circles and each circle is tangent to the x-axis, then we may shorten the radius of each circle until it is less than 1, preserving the point of tangency and keeping the new circle inside the original circle. We thus obtain a class C' of nonintersecting circles in one-to-one correspondence with the original set, with no radius exceeding 1. Let C_n' be those circles of C' with radius r satisfying $1/n\leqq r<1/(n-1)$. All the circles of C_n' which are tangent to the x-axis at points x, where $0\leqq x\leqq 1$, lie in a square of side 3 and hence are finite in number. Thus only a denumerable set of circles of C' can be tangent to the x-axis at points of the above interval. Since the irrational points in this interval are known to be nondenumerable, the class C' (therefore the class C also) must fail to fulfill the required condition.

Part II

1. The absolute value of each term is immediately seen to be $(100!)^2$ and an application of Wilson's theorem completes the proof.

2. By induction, $|X_{k+1}-1/A|=A^{2^k-1}|X_1-1/A|^{2^k}$. It follows that the sequence $\{X_k\}$, $k=1, 2, \cdots$, converges to $1/A$ if and only if $0<X_1<2/A$.

3. Since f is nonnegative and nonincreasing, $\int_0^1\int_0^1 f(x)f(y)(x-y)[f(x)-f(y)]\,dxdy \leqq 0$. Expanding, we have

$$\left[\int_0^1 xf^2(x)dx\right]\int_0^1 f(y)dy - \left[\int_0^1 xf(x)dx\right]\int_0^1 f^2(y)dy$$

$$-\left[\int_0^1 f^2(x)dx\right]\int_0^1 yf(y)dy + \left[\int_0^1 yf^2(y)dy\right]\int_0^1 f(x)dx$$

$$= 2\left[\int_0^1 xf^2(x)dx\right]\int_0^1 f(x)dx - 2\left[\int_0^1 xf(x)dx\right]\int_0^1 f^2(x)dx \leqq 0.$$

The conclusion is immediate.

4. Let A_n be the class of all finite sequences of ones and twos with n as the sum of the terms and let B_n be the class of all finite sequences with terms greater than 1 and sum n. Each sequence of A_n determines a sequence in B_{n+2} according to the following rule. Annex an additional term with value 2 as a final term and replace each maximal run of ones and the two which immediately follows the run by a single term with value equal to their sum. Thus 1, 1, 2, 2, 2, 1, 1, 1, 2, 1, 2, 1 becomes 4, 2, 2, 5, 3, 3. The correspondence is readily shown to be bi-unique and the proof follows.

5. Let C be the class of all subsets X of the given set such that $f(X)\subset X$. Put $T=\bigcap_{X\in C} X$. If $X\in C$, then $T\subset X$ and thus $f(T)\subset f(X)\subset X$. Thus $f(T)\subset T$ and therefore $T\in C$. Since $f(T)\subset T$, it follows that $f(f(T))\subset f(T)$ and therefore $f(T)\in C$. This implies $T\subset f(T)$, and the proof follows.

6. Let y and z be functions of x satisfying the respective equations $y''+h(x)y=0$ and $z''+g(x)z=0$ for $x\geqq 0$, where $h(x)\geqq g(x)$ for $x\geqq 0$ and where y and z satisfy the initial conditions $y'(0)=z'(0)=1$ and $y(0)=z(0)=0$. Let x_0 be the first zero of z to the right of the origin. Suppose y does not vanish in the interval $0<x\leqq x_0$. Then

$$0 = [z(yz' - y'z)/y]_0^{x_0} = \int_0^{x_0}\left[\frac{z(yz' - y'z)}{y}\right]' dx$$

$$= \int_0^{x_0}[(yz' - y'z)^2/y^2]dx + \int_0^{x_0} z^2[h(x) - g(x)]dx > 0,$$

since z does not vanish in the interior of the interval and h is assumed not to be identically equal to g on the interval. This contradiction shows that the first zero of y to the right of the origin is in the interval $0<x\leqq x_0$.

Take $g(x)=4/9$ and $h(x)=(x^2+4)/(x^2+9)$ and apply the above result. We find that the function $f(x)$ of the problem must vanish at some point x_0 such that $0<x_0\leqq 3\pi/2$. Next take $g(x)=(x^2+4)/(x^2+9)$ and $h(x)=53/63$. Then $h(x)>g(x)$ for $0\leqq x\leqq 3\pi/2$ and the above result shows that $\sin(x\sqrt{53/63})$ must vanish sooner than $f(x)$. The proof follows.

7. Let P be the regular polygonal disc of m sides, O its incenter, R the radius of the inscribed circle. Let n be a fixed positive integer and $C(n)$ the circle of center O and radius $R \sec(\pi/2mn)$. It will be shown that if $p_1, \cdots, p_n$ are any n points of $C(n)$, then there exists a rotation of $C(n)$ about the point O which will take the points p_i $(i=1, \cdots, n)$ into P. The numbers in the interval 0 to 2π correspond to the possible rotations of $C(n)$ about the point O. The numbers corresponding to the rotations which will take p_i out of P constitute m intervals in the interval 0 to 2π of total length not exceeding $2m(\pi/2mn)=\pi/n$. Thus the total length of the intervals corresponding to those rotations which take at least one of the points $p_1, p_2, \cdots, p_n$ out of the disc P is at most $n(\pi/n)=\pi$. It follows that there is at least one number on the interval 0 to 2π which corresponds to a rotation which takes none of the points p_i out of P, *i.e.*, there exists a rotation taking all the points p_i into P.

MATHEMATICAL NOTES

EDITED BY IVAN NIVEN, University of Oregon

Because of the large number of papers on hand, consideration of new papers for this department has been temporarily suspended.

PARTIAL FRACTIONS

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A genuine rational function $f(x)/g(x)$ can be decomposed into partial fractions by rationalizing and comparing coefficients. Whereas this method is rarely used in computational practice it leads to interesting systems of linear equations which it is worth while to discuss.

In the first principal case g is split into r coprime factors $\phi_1, \cdots, \phi_r$ with nonvanishing initial coefficients. We abbreviate $\phi_2\phi_3 \cdots \phi_r=\psi_1$ and decompose

$$f/g = \phi_1^*/\phi_1 + \psi_1^*/\psi_1 . \tag{1}$$

Given are $f(x)=\sum k_\iota x^\iota$, $\phi_1(x)=\sum a_\varkappa x^\varkappa$, and $\psi_1(x)=\sum b_\lambda x^\lambda$; assuming that $\phi_1^*(x)=\sum \alpha_\mu x^\mu$ and $\psi_1^*(x)=\sum \beta_\nu x^\nu$ the equations

$$\sum_{\mu+\lambda=\iota} \alpha_\mu b_\lambda + \sum_{\nu+\varkappa=\iota} \beta_\nu a_\varkappa = k_\iota, \qquad \iota = n-1, \cdots, 0, \tag{2}$$

(n the degree of g) for the unknown coefficients α_μ and β_ν follow immediately. The determinant of these equations is readily seen to be identical, except for transposition, with the resultant $R(\psi_1, \phi_1)$. Denoting the matrix of the (transposed) resultant R by $\mathbf{R}$, and introducing one-column matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{K}$ which contain the coefficients of ϕ_1^*, ψ_1^* and f, respectively, (2) can be written as

$$\mathbf{R}(\psi_1, \phi_1)\begin{pmatrix}\mathbf{B}\\ \mathbf{A}\end{pmatrix} = \mathbf{K}. \tag{3}$$

Setting $\psi_1 = \phi_2\psi_2$ we continue the decomposition:

$$\psi_1^*/\psi_1 = \phi_2^*/\phi_2 + \psi_2^*/\psi_2. \tag{4}$$

The procedure remains the same; with $\phi_2^* = \sum\gamma_\rho x^\rho$ and $\psi_2^* = \sum\delta_\sigma x^\sigma$ the system

$$\mathbf{R}(\psi_2, \phi_2)\begin{pmatrix}\mathbf{\Delta}\\ \mathbf{\Gamma}\end{pmatrix} = \mathbf{B} \tag{5}$$

for the new unknowns γ_ρ and δ_σ is obtained. (1) and (4) together lead to the decomposition

$$f/g = \phi_1^*/\phi_1 + \phi_2^*/\phi_2 + \psi_2^*/\psi_2 \tag{6}$$

into three partial fractions. In order to obtain it in a single step, without the detour through ψ_1^*, one may interpret (5) as a linear transformation which produces the β_ν from the γ_ρ and δ_σ. Extending this transformation so that it leaves the α_μ untransformed:

$$\begin{pmatrix}\mathbf{R}(\psi_2, \phi_2) & 0\\ 0 & E\end{pmatrix}\begin{bmatrix}\mathbf{\Delta}\\ \mathbf{\Gamma}\\ \mathbf{A}\end{bmatrix} = \begin{pmatrix}\mathbf{B}\\ \mathbf{A}\end{pmatrix} \tag{7}$$

(E the appropriate unit matrix), it serves to eliminate $\mathbf{B}$ from (3) to give the system

$$\mathbf{R}(\psi_1, \phi_1)\begin{pmatrix}\mathbf{R}(\psi_2, \phi_2) & 0\\ 0 & E\end{pmatrix}\begin{bmatrix}\mathbf{\Delta}\\ \mathbf{\Gamma}\\ \mathbf{A}\end{bmatrix} = \mathbf{K}$$

for the coefficients required in (6).

Successive application of transformations analogous to (7) finally leads to a system with the matrix

$$\mathbf{R}(\psi_1, \phi_1)\begin{pmatrix}\mathbf{R}(\psi_2, \phi_2) & 0\\ 0 & E\end{pmatrix}\cdots\begin{pmatrix}\mathbf{R}(\phi_r, \phi_{r-1}) & 0\\ 0 & E\end{pmatrix}$$

for the unknown coefficients in the decomposition

$$f/g = \phi_1^*/\phi_1 + \cdots + \phi_r^*/\phi_r. \tag{8}$$

The determinant of this system is

$$R(\psi_1, \phi_1)R(\psi_2, \phi_2) \cdots R(\phi_r, \phi_{r-1}) = \prod_{i>k} R(\phi_i, \phi_k) \tag{9}$$

which cannot vanish; thus the solutions are unique, as was to be expected. However, one can also invert the argument. Assuming the possibility of a decomposition in the form (8), the comparison of coefficients leads to (9) which shows that this decomposition is unique if and only if $\phi_1, \cdots, \phi_r$ are coprime in pairs and have nonvanishing initial coefficients. The existence and uniqueness of the familiar decomposition is thus established in another way.

The other principal case arises when $g(x)$ is the mth power of a polynomial $\phi(x)$ of degree n. Here

$$f/g = \phi_m^*/\phi^m + \cdots + \phi_1^*/\phi \tag{10}$$

with numerators ϕ_μ^* of degrees $<n$. Given are

$$\phi(x) = \sum_{\lambda=0}^{n} a_\lambda x^\lambda \quad \text{and} \quad f(x) = \sum_{\iota=0}^{mn-1} k_\iota x^\iota,$$

where $a_n \neq 0$; we then set with unknown coefficients

$$\phi_\mu^*(x) = \sum_{\nu=0}^{n-1} \alpha_{\mu,\nu} x^\nu,$$

abbreviate the polynomial development of $[\phi(x)]^\rho$ by $\phi^\rho = \sum C(\rho, \tau)x^\tau$, substitute in (10), make rational and find

$$\sum_{\iota=0}^{mn-1} k_\iota x^\iota = \sum_{\rho=0}^{m-1} \phi_{m-\rho}^* \phi^\rho = \sum_{\rho=0}^{m-1} \sum_{\nu=0}^{n-1} \sum_{\tau=0}^{\rho n} \alpha_{m-\rho,\nu} C(\rho, \tau) x^{\nu+\tau}.$$

In the triple sum we first interchange to $\sum_{\nu=0}^{n-1} \sum_{\tau=0}^{(m-1)n} \sum_{\rho=[\tau/n]}^{m-1}$ where the summation over ρ begins at $[\tau/n]$, the "next integer after τ/n":

$$[\tau/n] \text{ integer and } \tau/n \leqq [\tau/n] < \tau/n + 1; \tag{11}$$

furthermore we introduce the independent index of summation $\iota = \nu + \tau$ and obtain $\sum_{\iota=0}^{mn-1} \sum_{(A)} \sum_{\rho=[\tau/n]}^{m-1}$ where the middle sum has to be formed according as

$$\begin{cases} \nu + \tau = \iota \\ 0 \leqq \nu \leqq n-1 \\ 0 \leqq \tau \leqq (m-1)n. \end{cases} \tag{A}$$

Now the comparison of coefficients is possible and yields

$$\sum_{(A)} \sum_{\rho=[\tau/n]}^{m-1} \alpha_{m-\rho,\nu} C(\rho, \tau) = k_\iota, \qquad \iota = 0, \cdots, mn-1. \tag{12}$$

These mn equations are best ordered according to increasing ι; thus every equation gets the number $\iota+1$. Moreover, in each equation the unknowns $\alpha_{\mu\nu}$ shall be put into the order

$$\alpha_{m,0} \cdots \alpha_{m,n-1}\alpha_{m-1,0} \cdots \alpha_{m-1,n-1} \cdots \alpha_{1,0} \cdots \alpha_{1,n-1};$$

thus every unknown gets the number $\rho n+\nu+1$. If the system is written down in this way its matrix contains only zeros below, but no zeros in, the principal diagonal.

Proof in two steps. Firstly, equation number $\iota+1$ contains only unknowns with numbers $\geqq\iota+1$. In order to write down this equation one must begin with the diophantine equation $\nu+\tau=\iota$, according to (A). The unknowns pertaining to the solutions of the latter have the numbers $\rho n+\nu+1$, with $\rho\geqq[\tau/n]$ according to (11). Hence $\rho n-\tau\geqq 0$, or $\rho n+\nu+1=\rho n+(\iota-\tau)+1\geqq\iota+1$.

Secondly, for the unknown number $\iota+1$ to occur in equation number $\iota+1$ it is sufficient that among the mentioned solutions of $\nu+\tau=\iota$ there be one pair ν_0, τ_0 with $\tau_0\equiv 0 \bmod n$, because in this case $\tau_0=\rho_0 n$ with integral ρ_0, or $\nu_0+\rho_0 n=\iota$, or $\rho_0 n+\nu_0+1=\iota+1$. This condition of sufficiency is, however, fulfilled for every ι. The congruence $\tau=\iota-\nu\equiv 0 \bmod n$ has a unique solution ν_0 for arbitrary ι if and only if ν_0 can be taken from a complete system of residues mod n, and just this is guaranteed by (A). More explicitly, it is seen that

$$\begin{aligned}
&\text{to } \iota = 0, \cdots, n-1 && \text{belongs } \tau_0 = 0,\\
&\text{to } \iota = n, \cdots, 2n-1 && \text{belongs } \tau_0 = n,\\
&\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\text{to } \iota = (m-1)n, \cdots, mn-1 && \text{belongs } \tau_0 = (m-1)n.
\end{aligned}$$

Thus there are m groups of n equations; in each group τ_0 and hence $\rho_0=\tau_0/n$ are the same, so that all equations of a group also begin with the same coefficient $C(\rho_0, \tau_0)=C(\rho_0, \rho_0 n)=a_n^{\rho_0}$.

Under these circumstances the determinant of the system (12) becomes

$$D = (a_n^{0})^{n}(a_n^{1})^{n} \cdots (a_n^{m-1})^{n} = a_n^{(m-1)mn/2}. \tag{13}$$

Since this cannot vanish, the system (12) has always a unique solution. As above, one can invert the argument. Assuming the validity of (10) the result (13) shows that the decomposition exists and is unique if and only if $a_n\neq 0$.

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A RELATIONSHIP BETWEEN SEMI-MAGIC SQUARES AND PERMUTATION MATRICES

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A semi-magic square is an n by n matrix A over a field F for which the sum of the n elements in any row or column is a constant $a=S(A)$. It was shown in [1] that the semi-magic squares of order n under the usual matrix addition and multiplication form an algebra $\mathfrak{M}$ over F.

THEOREM. *If* Π *is the set of permutation matrices and* $\{\Pi\}$ *is the algebra generated by* Π *over* F, *then* $\mathfrak{M}=\{\Pi\}$.

Proof. Let $\mathfrak{R}$ be the set of matrices of the form aI where a is in F and let $\mathfrak{N}$ be the set of matrices for which $S(A)=0$. It was shown in [1] that the matrices

$$N_{ij}=\begin{array}{c} \\ \\ \\ \\ i \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & 0\cdots 0 & \overset{j}{-1} & 0\cdots 0 \\ 0 & & 0 & \\ \vdots & & \vdots & \vdots \\ 0 & & 0 & \\ -1 & 0\cdots 0 & 1 & 0\cdots 0 \\ 0 & & 0 & \\ \vdots & & \vdots & \\ 0 & \cdots & 0 & \cdots 0 \end{bmatrix}$$

form a basis of $\mathfrak{N}$ over F. Considering $\mathfrak{M}$ as a vector space over F, $\mathfrak{M}=\mathfrak{R}+\mathfrak{N}$: if A is in $\mathfrak{M}$ where $S(A)=a$, then $A=aI+(A-aI)$ and $A-aI$ is obviously in $\mathfrak{N}$.

Now suppose P is in Π, then $S(P)=1$. Thus P is in $\mathfrak{M}$ and therefore $\Pi\subset\mathfrak{M}$. Since $\mathfrak{M}$ is an algebra the sums and products of any matrices in $\mathfrak{M}$ are again in $\mathfrak{M}$, and since $\Pi\subset\mathfrak{M}$, $\{\Pi\}\subset\mathfrak{M}$. Conversely to show that $\mathfrak{M}\subset\{\Pi\}$ it will be sufficient to show that the basis matrices N_{ij} and I can be written as sums of products of permutation matrices. Since I is in $\{\Pi\}$, we need only show the N_{ij} are in $\{\Pi\}$. Now $N_{ij}=PN_{nn}Q$ for suitable permutation matrices P and Q *i.e.* P interchanges the nth row of N_{nn} and the ith row of N_{nn}, and Q interchanges the nth column of N_{nn} and the jth column of N_{nn}. However

$$N_{nn}=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}-\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & 1 & \\ 1 & 0\cdots & 0 & 0 \end{bmatrix}=I-P_1$$

is in $\{\Pi\}$ since I and P_1 are in Π. Therefore $N_{ij}=P(I-P_1)Q=PQ-PP_1Q$ is in $\{\Pi\}$ and hence $\mathfrak{M}\subset\{\Pi\}$.

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PARTIAL ORDER AND THE SUCCESSOR FUNCTION

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The successor function; *i.e.*, the function which assigns to each element of a well-ordered set the "next element", is usually defined for just well-ordered sets. The purpose here is to define this function for partially ordered sets* and to prove the well-orderedness of these sets assuming a modified mathematical induction principle. For this we need:

DEFINITION 1. *If S is a partially ordered set and $s\in S$, then the (immediate) successor of s is an element of S which properly includes s and is included in all elements which properly include s, if such exists, or is s in case s is a maximal element of S. "s'" will denote the successor of s whenever it exists.*

DEFINITION 2. *A partially ordered set S with a least element satisfies the mathematical induction principle if and only if every subset S_1 of S, containing the least element of S and containing the successor (whenever it exists) of every element of S_1, is equal to S.*

THEOREM. *If S is a partially ordered set with a least element s_0 and satisfies the mathematical induction principle, then S is well-ordered.*

Proof. Assume the hypothesis, and also that there is a nonnull subset R of S without a least element. Let S_1 be the set of all elements of S which "precede" R; *i.e.*, which are properly included in each element of R. ($s_0\in S_1$ since R has no least element and s_0 is the least element in S.)

The proof is by cases:

Case 1. Each element of S_1 has a successor in S. $s_0\in S_1$, and if $s\in S_1$ then $s'\in S_1$ (otherwise s' would be the least element of R). Hence $S_1=S$ and so R is null, contrary to assumption.

Case 2. There exists an element x of S_1 which has no successor in S. By letting $S_1'=S/x\cup\{x\}$ (where S/x is the set of all elements of S properly included in x), then $s_0\in S_1'$, and if $s\in S_1'$ then $s'\in S_1'$ (whenever s' exists). Therefore $S_1'=S$ and R is null, a contradiction.

The two cases, being exhaustive, establish the theorem.

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* "Partially ordered set," "least element," "maximal element," and "well-ordered set" are defined in G. Birkhoff, Lattice Theory, American Mathematical Society, Colloquium Publications, vol. 25 Revised ed., 1949. See pp. 1, 7, 7. and 32, respectively.

ON A GEOMETRICAL THEOREM

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Given a triangle $A_1A_2A_3$ and a line through any point P in the plane, meeting the sides of the triangle at X_1, X_2, X_3 respectively. Let A_iP $(i=1, 2, 3)$ meet the circumcircle at R_i. The three lines X_iR_i are concurrent at a point E on the circumcircle [1, p. 238]. We propose to show how to associate a definite point E with a definite line on P and *vice versa*.

We shall use the circumcircle $A_1A_2A_3$ as the base circle and let the coordinates of the vertices A_i $(i=1, 2, 3)$ be turns t_i, *i.e.* complex numbers with unit modulus. Let T be any other point on the circumcircle with coordinate T, and P a fixed point not on the circle with cordinate p. We choose as the line on P, a line perpendicular to the image line of T [2, p. 422].

The equation of this line is $Tx+\sigma_3\bar{x}=pT+\sigma_3\bar{p}$, where $\sigma_1=t_1+t_2+t_3$, $\sigma_2=t_2t_3+t_3t_1+t_1t_2$, $\sigma_3=t_1t_2t_3$. The coordinates of the points X_i and R_i are readily found to be $(T-t_i)x_i=pT+\sigma_3\bar{p}-(t_j+t_k)\sigma_3$, $(1-\bar{p}t_i)r_i=p-t_i$. The three lines cut the circumcircle of $A_1A_2A_3$ at

$$E=\frac{\sigma_3[p-\sigma_1+\sigma_2\bar{p}-\sigma_3\bar{p}^2+(1-p\bar{p})T]}{T(p^2-\sigma_1p+\sigma_2-\sigma_3\bar{p})-(1-p\bar{p})\sigma_3}=\frac{q-T}{1-\bar{q}T},$$

where q is the coordinate of Q, the isogonal conjugate point of P as to the triangle $A_1A_2A_3$ [2, p. 425]. From the form of the coordinate of E, it is evident that the points T, Q, E are collinear.

Hence, given a triangle $A_1A_2A_3$ and a point P, to determine what transversal through P will produce a chosen E on the circumcircle we proceed as follows. Join E to Q, the isogonal conjugate point of P as to $A_1A_2A_3$, produce EQ cutting the circle $A_1A_2A_3$ at T. Draw the image line of T, and the line through P perpendicular to this image line is the required transversal.

On the other hand given the transversal through P to determine the point E we proceed thus. Draw a line through H, the orthocenter of $A_1A_2A_3$, perpendicular to the transversal. This line will be the line of images of some point T on the circumcircle, and T can be located [2, p. 423]. The line joining T to Q, the isogonal conjugate of P as to $A_1A_2A_3$, cuts the circumcircle at the required point E.

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AN APPLICATION OF THE GAUSS MULTIPLICATION THEOREM

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In a previous note (this MONTHLY, vol. 62, 1955, p. 120), S. K. L. Rao derives Legendre's duplication formula by means of Mellin transforms. On attempting to extend this method to a proof of the Gauss multiplication theorem, one is led to the evaluation of some interesting integrals.

The Gauss multiplication theorem [1] states that

$$\Gamma(s)\Gamma(s+1/n)\cdots\Gamma(s+(n-1)/n) = (2\pi)^{(n-1)/2}n^{(1/2)-ns}\Gamma(ns). \tag{1}$$

For $n=3$, this reduces to the triplication formula

$$\Gamma(s)\Gamma(s+1/3)\Gamma(s+2/3) = (2\pi)3^{1/2-3s}\Gamma(3s). \tag{2}$$

If we now let

$$F_1(x) = e^{-x}, \quad F_2(x) = x^{1/3}e^{-x}, \quad \text{and} \quad F_3(x) = x^{2/3}e^{-x}, \tag{3}$$

then the corresponding Mellin transforms are

$$MF_1(x) = \Gamma(s), \quad MF_2(x) = \Gamma(s+1/3), \quad \text{and} \quad MF_3(x) = \Gamma(s+2/3), \tag{4}$$

where by definition $MF(x)=\int_0^\infty F(x)x^{s-1}dx$. By the Faltung theorem [2]

$$MF(x)\cdot MG(x) = M\int_0^\infty F(x/u)G(u)u^{-1}du. \tag{5}$$

Consequently,

$$MF_1(x)\cdot MF_2(x) = M\int_0^\infty e^{-x/u-u}u^{-2/3}du = MG(x). \tag{6}$$

We now evaluate the following generalization of $G(x)$:

$$I(x) = \int_0^\infty e^{-[t^n+xt^{-n}]}dt, \qquad (\text{for } n=3,\ G(x)=3I(x)). \tag{7}$$

By integration by parts and differentiation through the integral sign, it follows that

$$\left[D^2 + \frac{n-1}{nx}D - \frac{1}{x}\right]I(x) = 0. \tag{8}$$

The solution to this latter equation [3] is given by

$$I(x) = x^{1/2n}\left[AI_{1/n}(2\sqrt{x}) + BK_{1/n}(2\sqrt{x})\right]. \tag{9}$$

Since [4]

(10) $I_p(x) \sim \dfrac{e^x}{\sqrt{2\pi x}}, \qquad K_p(x) \sim e^{-x}\sqrt{\dfrac{\pi}{2x}}, \quad \text{and} \quad \lim_{x\to 0} \dfrac{K_p(x)}{2^{p-1}\Gamma(p)x^{-p}} = 1,$

$$I(x) = (2/n)x^{1/2n}K_{1/n}(2\sqrt{x}).$$

For $n=2$, we get a well known result [5] $I(x)=\sqrt{\pi}/2e^{-2\sqrt{x}}$.

By application of the Faltung theorem again, we get

(11) $$MF_3(x)\cdot MG(x) = 2Mx^{2/3}\int_0^\infty e^{-x/u}K_{1/3}(2\sqrt{u})u^{-3/2}du.$$

Since $MF_3(x)\cdot MG(x)=(2\pi)3^{1/2-3s}\Gamma(3s)$ (by (2) and (5)) $=M((2\pi/\sqrt{3})e^{-3x^{1/3}})$, it follows that

(12) $$\int_0^\infty e^{-x/u}K_{1/3}(2\sqrt{u})u^{-3/2}du = (\pi/\sqrt{3})x^{-2/3}e^{-3x^{1/3}}.$$

Similarly, by different pairings of the functions F_1, F_2, F_3, we obtain

(13) $$\int_0^\infty e^{-x/u}K_{2/3}(2\sqrt{u})u^{-1}du = (\pi/\sqrt{3})x^{-1/3}e^{-3x^{1/3}},$$

(14) $$\int_0^\infty e^{-x/u}K_{1/3}(2\sqrt{u})u^{-1/2}du = (\pi/\sqrt{3})e^{-3x^{1/3}}.$$

[(12) follows directly from (14) by differentiating through the integral sign]. These last three integrals give rise to the following Laplace transforms:

(15) $$p\int_0^\infty e^{-pt}K_{1/3}(2t^{-1/2})t^{-1/2}dt = (\pi/\sqrt{3})p^{1/3}e^{-3p^{1/3}},$$

(16) $$p\int_0^\infty e^{-pt}K_{2/3}(2t^{-1/2})t^{-1}dt = (\pi/\sqrt{3})p^{2/3}e^{-3p^{1/3}},$$

(17) $$p\int_0^\infty e^{-pt}K_{1/3}(2t^{-1/2})t^{-3/2}dt = (\pi/\sqrt{3})pe^{-3p^{1/3}}.$$

Proceeding in a similar manner from the Gauss multiplication theorem for $n=4$, we also find that

(18) $$\int_0^\infty K_{1/4}(2\sqrt{x/u})K_{1/4}(2\sqrt{u})u^{-1/2}du = (1/8)x^{-1/8}(2\pi)^{3/2}e^{-4x^{1/4}},$$

(19) $$\int_0^\infty K_{3/4}(2\sqrt{x/u})K_{1/4}(2\sqrt{u})u^{-1}du = (1/8)x^{-3/8}(2\pi)^{3/2}e^{-4x^{1/4}},$$

and

$$(20) \qquad \int_0^\infty K_{1/2}(2\sqrt{x/u})K_{1/2}(2\sqrt{u})u^{-3/4}du = (1/8)x^{-1/4}(2\pi)^{3/2}e^{-4x^{1/4}}.$$

(20) corresponds to the known integral $\int_0^\infty e^{-x^2-a^2/x^2}dx = (1/2)e^{-2a}\sqrt{\pi}$, and also to (9) for $n=2$.

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CLASSROOM NOTES

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FROM FOURIER SERIES TO FOURIER INTEGRAL

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It is natural to think of the Fourier integral as the extension of the Fourier series to the case where the range is infinite. Heuristic arguments (as in [1], [2], [3], [4]) are sometimes given to show the transition from series to integral. The development that follows is offered as an alternative method of extending the series to the integral. It is most useful for the case when one has already been discussing convergence of the Fourier series, and hence has equation (3) below for the sum of the first N terms of the Fourier series.

Let

$$(1) \qquad S = a_0/2 + \sum_1^\infty \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$(2) \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos \frac{n\pi t}{L}\, dt, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin \frac{n\pi t}{L}\, dt.$$

It is well known (see, for example [1], p. 386) that the sum of the first N

terms of the above series may be written

$$S_N = \frac{1}{2L}\int_{-L}^{L} f(t)\,\frac{\sin\left[(N+1/2)(t-x)\pi/L\right]}{\sin\left[(t-x)\pi/2L\right]}\,dt. \tag{3}$$

Hence

$$S_N = \frac{1}{2L}\int_{-L}^{L}\frac{f(t)(t-x)}{\sin\left[(t-x)\pi/2L\right]}\int_{0}^{(N+1/2)\pi/L}\cos\left[(t-x)q\right]dq\,dt \tag{4}$$

$$= \frac{1}{\pi}\int_{0}^{(N+1/2)\pi/L}\int_{-L}^{L} f(t)\,\frac{(t-x)\pi/2L}{\sin\left[(t-x)\pi/2L\right]}\cos\left[(t-x)q\right]dt\,dq. \tag{5}$$

This is now an expression for the sum of N terms of a Fourier series for $f(x)$ in the interval $(-L, L)$. It will in general fit the function at just N points, with the average spacing between the abscissae of these points being $2L/N$. To improve the fit, one must let $N\to\infty$. If L is fixed, and $f(x)$ satisfies the Dirichlet conditions, then S_N will converge to $[f(x+0)+f(x-0)]/2$. However, if L is increased as N is increased, then it is necessary to relate the rates at which L and N are increased so as to have $2L/N\to 0$ as L and N are increased. This will be the case, for example, if $L=N^k$, where $0<k<1$. Upon setting $L=N^k$ in (5), and letting $N\to\infty$, one obtains

$$S = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos q(t-x)\,dt\,dq, \tag{6}$$

which is the Fourier integral.

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AN INTERESTING FOURTH-ORDER DIFFERENTIAL SYSTEM

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Although the theory of linear differential equations with constant coefficients is well known, most of the examples and problems given in texts have characteristic equations which are either first- or second-order or are easily factorable. In this note an important subset of the set of all fourth-order differential systems will be described. The solution of members of this subset can be obtained by the solution of at most two quadratic equations instead of the usual quartic equation.

The most general linear homogeneous fourth-order system of differential equations with constant coefficients may be written in the form

$$x'' = C_1x' + C_2x + C_3y' + C_4y, \tag{1}$$

$$y'' = C_5x' + C_6x + C_7y' + C_8y, \tag{2}$$

where C_i are constants, and primes denote derivatives with respect to the independent variable t. If (2) is multiplied by i and added to (1), and the complex variable $\xi = x + iy$ is introduced, the following complex equation may be written

$$\xi'' + A\xi' + B\xi + C\bar{\xi}' + D\bar{\xi} = 0, \tag{3}$$

where the complex coefficients, A, B, C, D are linear combinations of the C_i. For the special case of $C = D = 0$, (3) can be easily solved with solution

$$\xi = K_1e^{i\phi_1} + K_2e^{i\phi_2}, \tag{4}$$

where

$$K_j = K_{j_0}e^{\alpha_j t},$$
$$\phi_j = \phi_{j_0} + \phi_j' t,$$
$$\alpha_j + i\phi_j' = \tfrac{1}{2}[-A \pm \sqrt{A^2 - 4B}].$$

This solution curve for nonvanishing ϕ_j' is swept out by the sum of two two-dimensional vectors with exponentially damped amplitudes K_j and constant angular velocities ϕ_j'. Such curves are called damped epicycles. Because of the simplicity of this special form it is of some interest to determine the conditions under which it is possible to transform (3) to this special form by means of a nonsingular linear transformation. Those systems which satisfy these conditions will be said to be members of the epicyclic subset of the set of fourth-order systems.

A nonsingular linear transformation can be written either in the form

$$\lambda = me^{i\theta}(\xi + s\bar{\xi}), \tag{5}$$

where $m(1 - s\bar{s}) \neq 0$ and m is real, or in a similar form for $\bar{\lambda}$. Without loss of generality we restrict ourselves to (5). The inverse of (5) is

$$\xi = \frac{e^{-i\theta}\lambda - se^{i\theta}\bar{\lambda}}{m(1 - s\bar{s})}. \tag{6}$$

Substituting (6) in (3)

$$\lambda'' + (A - C\bar{s})\lambda' + (B - D\bar{s})\lambda - e^{2i\theta}[s\bar{\lambda}'' + (As - C)\bar{\lambda}' + (Bs - D)\bar{\lambda}] = 0. \tag{7}$$

If $\bar{\lambda}''$ is eliminated between (7) and its conjugate,

$$\begin{aligned}(1 - s\bar{s})\lambda'' &+ (A - C\bar{s} + \overline{C}s - \overline{A}s\bar{s})\lambda' \\ &+ (B - D\bar{s} + \overline{D}s - \overline{B}s\bar{s})\lambda - e^{2i\theta}\{[\overline{C}s^2 + (A - \overline{A})s - C]\bar{\lambda}' \\ &+ [\overline{D}s^2 + (B - \overline{B})s - D]\bar{\lambda}\}^* = 0.\end{aligned} \tag{8}$$

The requirement that (3) belong to the epicyclic subset means that it must be possible to select m, θ, and s so that the coefficients of $\bar{\lambda}'$ and $\bar{\lambda}$ vanish. For this purpose, we arbitrarily select m and θ to be unity and zero respectively. With the definitions $A=A_1+iA_2$, $B=B_1+iB_2$, $C=|C|e^{i\sigma}$, and $D=|D|e^{i\omega}$, it can be seen that s must simultaneously satisfy the following pair of equations:

$$|C|s^2+2iA_2se^{i\sigma}-|C|e^{2i\sigma}=0, \tag{9}$$

$$|D|s^2+2iB_2se^{i\omega}-|D|e^{2i\omega}=0. \tag{10}$$

If neither C nor D vanishes, the solutions of (9–10) are

$$s=\left[\frac{A_2}{|C|}\pm\sqrt{\left|\frac{A_2}{C}\right|^2-1}\right]e^{i(\sigma-(\pi/2))}, \tag{11}$$

$$s=\left[\frac{B_2}{|D|}\pm\sqrt{\left|\frac{B_2}{D}\right|^2-1}\right]e^{i(\omega-(\pi/2))}. \tag{12}$$

Since $|s|$ is unity when either $|A_2/C|$ or $|B_2/D|$ is less than or equal to one, and this contradicts our assumption of nonsingularity,

$$\left|\frac{A_2}{C}\right|>1, \qquad \left|\frac{B_2}{D}\right|>1.$$

Under these restrictions, s can be a root of both (9) and (10) if and only if

$$\frac{A_2}{C}=\frac{B_2}{D}. \tag{13}$$

From these results the following theorem can be derived.

THEOREM. *Equation* (2) *is a member of the epicyclic subset if one of the following conditions is satisfied*:

(a) $C=D=0$

(b) $C=0, \quad A_2=0, \quad |B_2/D|>1$

(c) $D=0, \quad B_2=0, \quad |A_2/C|>1$

(d) $A_2/C=B_2/D, \quad |A_2/C|>1.$

For the trivial case (a), s is zero; for the other cases it is fixed by either (11) or (12).

It should be emphasized that (13) represents two conditions on the eight real constants in (3) and, therefore, the epicyclic subset contains a number of important physical problems. For example, the small amplitude angular motion of a top or a symmetric missile in flight can be described by a member of the epicyclic subset.

A DEVELOPMENT OF LOGARITHMS USING THE FUNCTION CONCEPT

C. L. SEEBECK, JR. and JOHN W. JEWETT, University of Alabama and University of Georgia

The traditional approach to the logarithmic function in freshman college or high school courses is that of the inverse of the exponential function. The exponential function itself is seldom defined for the entire real line and is a difficult concept for immature students for all values except the positive integer. Any experienced teacher has encountered the difficulties inherent in obtaining real understanding with this approach. In keeping with the modern trend to devote more time to the modern concept of function in both college freshman and high school courses, the authors present a development of the logarithmic function which they believe to be direct and simple, and one which serves as an excellent example of the function concept.

DEFINITION. *Let R^+ denote the set of positive real numbers. The logarithm function is a function* $\log: R^+ \Rightarrow R$ *satisfying the following assumptions for all a and b in R^+.*

(1) $$\log (a \cdot b) = \log (a) + \log (b),$$

(2) $$\log (10) = 1,$$

(3) *If* $a > 1$, *then* $\log (a) > 0$.

We insist on retaining the parentheses in $\log (a)$ to emphasize the fact that $\log (a)$ is the value of the logarithm function at a.

Next we prove, or ask the students to prove as exercises the following simple theorems:

THEOREM 1. $\log (1) = 0$.

Proof. $\log (1) = \log (1 \cdot 1) = \log (1) + \log (1) = 2 \log (1)$. But the only number that is twice itself is the number zero.

THEOREM 2. $\log (1/a) = -\log (a)$.

Proof. $\log (1) = \log (a/a) = \log (a \cdot 1/a) = \log (a) + \log (1/a)$.
But $\log (1) = 0$, so $\log (1/a)$ must be the negative of $\log (a)$.

THEOREM 3. $\log (a/b) = \log (a) - \log (b)$.

Proof. $\log (a/b) = \log (a \cdot 1/b) = \log (a) + \log (1/b) = \log (a) - \log (b)$.

THEOREM 4. *If m is an integer, then* $\log (a^m) = m \log (a)$.

Proof. $\log (a^m) = \log (a \cdot a \cdot \cdots a)$ with m factors. Using assumption (1) $\log (a^m) = \log (a) + \log (a) + \cdots + \log (a)$ with m terms. Hence $\log (a^m) = m \log (a)$.

COROLLARY. *If m is an integer,* $\log (10^m) = m$.

THEOREM 5. *If n is an integer,* $\log (a^{1/n}) = (1/n) \log (a)$.

Proof. Let $N=a^{1/n}$. Then $N^n=a$ and $n \log (N)=\log (a)$. Hence $\log N = (1/n) \log (a)$.

COROLLARY 1. *If m and n are integers,* $\log (a^{m/n})=(m/n) \log (a)$.

COROLLARY 2. *If m and n are integers,* $\log (10^{m/n})=m/n$.

THEOREM 6. *If $a<b$, then* $\log (a)<\log(b)$.

Proof. If $a<b$ then $b=ac$, where $c>1$. Hence $\log (b)=\log (a)+\log (c)$. But by Assumption (3), $\log (c)>0$. Hence $\log (a)<\log (b)$.

COROLLARY. *If $a<1$, then* $\log (a)<0$.

We now show the student that any positive number can be written as $10^n \cdot p$, where n is an integer and p a number between one and ten. Moreover, since $\log (10^n \cdot p)=n+\log p$, the problem of finding the values of the logarithm function is reduced to the problem of finding the values between one and ten. We then pass out an ordinary four place logarithm table which we label, "Values of $\log (a)$ for a between one and ten."

Interpolation is justified as a reasonable approximation for $a<x<b$, by noting that $\log (a)<\log (x)<\log (b)$, for $b-a$ arbitrarily small. We call n the "whole number part" and $\log (p)$ the "fractional part" of the logarithm. We proceed to the usual computational work using logarithms.

The proof of the uniqueness of the logarithmic function given below is beyond the scope of an elementary course.

THEOREM. *The only function satisfying assumptions* (1) *to* (3) *above is the usual logarithm function to the base* 10.

Proof. We need a result quoted in [1, p. 16] which, when specialized to the real line, reads as follows:

Let f and g be real valued monotone functions defined on R^+, let X be a subset of R^+ on which f and g agree and let Y_0 be $f(X)$. A sufficient condition that $f=g$ is that Y_0 intersect every set of the form $(y: u<y<v)$, where u and v are real numbers.

Since we have let *log* denote a function satisfying our assumptions (1) and (3), let the ordinary logarithm function to the base ten be written *Log*. Let X_0 denote the set of all positive real numbers which can be written as rational powers of ten.

By the second corollary to Theorem 5, $\log (a)=\text{Log } (a)$ on X. But this corollary also implies that log maps X onto the set of all rational numbers. The set of all rational numbers intersects every open interval of real numbers. Hence the result quoted from [1] implies that $\log (a)=\text{Log } (a)$ for all positive real numbers.

Reference

1. J. L. Kelley, General Topology, New York, 1955.

THREE THEOREMS ON PERMUTATIONS

A. A. MULLIN, Massachusetts Institute of Technology

As a corollary to the binomial theorem it is often proved that

$$\sum_{r=0}^{n} {}_nC_r \equiv 2^n, \quad \text{where} \quad {}_nC_r \equiv \frac{n!}{(n-r)!r!}, \tag{1}$$

and where n is a nonnegative integer ($0! \equiv 1$).

Also, if $\Psi(n)$ is defined by

$$\Psi(n) \equiv \sum_{r=0}^{n} {}_nC_r, \tag{2}$$

and if $\Psi(0) \equiv 1$, then a canonic generator for Ψ is

$$\Psi(n+1) \equiv 2 \cdot \Psi(n). \tag{3}$$

Now, consider the function defined by

$$\Phi(n) \equiv \sum_{r=0}^{n} {}_nP_r, \quad \text{where} \quad {}_nP_r \equiv \frac{n!}{(n-r)!}, \tag{4}$$

and where n is a nonnegative integer ($0! \equiv 1$), and $\Phi(0) \equiv 1$.

THEOREM 1. *A canonic generator for Φ is*

$$\Phi(n+1) \equiv \{(n+1)\Phi(n)\} + 1. \tag{5}$$

Proof. Consider the right-hand side (*rhs*) of (5),

$$\begin{aligned} &\{(n+1)[1 + {}_nP_1 + \cdots + {}_nP_n)\} + 1 \\ &= \{(n+1) + (n+1)_nP_1 + \cdots + (n+1)_nP_n\} + 1 \\ &= \{{}_{n+1}P_1 + {}_{n+1}P_2 + \cdots + {}_{n+1}P_{n+1}\} + {}_{n+1}P_0; \end{aligned}$$

thus, $rhs \equiv \sum_{r=0}^{n+1} {}_nP_r$; but, by definition, the left-hand side (*lhs*) of (5) is $\sum_{r=0}^{n+1} {}_nP_r$ or, $lhs = rhs$.

THEOREM 2. *An asymptotic expression for $\Phi(n)$ is*

$$\Phi(n) \sim (n!) \cdot e, \quad \textit{where} \quad \begin{matrix} n \gg 1 \\ e = 2.718\cdots. \end{matrix} \tag{6}$$

Proof. $\Phi(n) \equiv {}_nP_0 + {}_nP_1 + \cdots + {}_nP_n$ factor out ${}_nP_n \equiv n!$ from the *rhs*, yielding

$$\Phi(n) \equiv n!\left[\frac{1}{n!} + \frac{1}{(n-1)!} + \cdots + 1 + 1\right],$$

but for $n \gg 1$ the bracketed expression approaches e, or $\Phi(n) \sim (n!) \cdot e$, where $e = 2.718\cdots$.

Note. It should not be too surprising that (6) is the asymptotic form of $\Phi(n)$ in view of the form of (5), which implies a factorial function when n is large.

LEMMA. *An asymptotic expression for* $n!$ *is*

$$n! \sim n^n e^{-n}\sqrt{2\pi n}. \tag{7}$$ (*Stirling's Formula*)

THEOREM 3. *An asymptotic expression for* $\Phi(n)$ *is*

$$\Phi(n) \sim n^n e^{1-n}\sqrt{2\pi n}. \tag{8}$$

Proof. Substitute Lemma into Theorem 2.

Tabulations:

$$\Phi(0) = 1,$$
$$\Phi(1) = 2,$$
$$\Phi(2) = 5,$$
$$\Phi(3) = 16,$$
$$\Phi(4) = 65,$$
$$\Phi(5) = 326,$$
$$\Phi(6) = 1957,$$
$$\Phi(10) = 9.8641 \times 10^6.$$

Using Theorem 3 and a slide rule yields $\Phi(10) = 9.79 \times 10^6$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1286. *Proposed by F. S. Stancliff, Springfield, Ohio*

Establish the identity

$$\sum_{r=0}^{n-2} (2^r - 1)^2 + [3(2^{n-1}) - 1]^2 = \sum_{r=2}^{n} (2^r + 1)^2 + (2^n - 4)^2,$$

thus obtaining a general method for finding two equal sums of n distinct squares.

E 1287. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that the radical center P of the circles inscribed in the equilateral triangles BCA', CAB', ABC' constructed exteriorly (or interiorly) on the sides of a triangle ABC having centroid G divides the distance from G to the point of concurrence Q of the lines AA', BB', CC' in the ratio $GP/GQ=1/4$.

E 1288. *Proposed by S. H. Kimball, University of Maine*

The number of odd binomial coefficients in any finite binomial expansion is a power of 2 (Putnam Mathematical Competition, this MONTHLY [1957, p. 24]). Prove that the power of 2 is the number of 1's in the binary scale expression for n in $(x+y)^n$.

E 1289. *Proposed by Marvin Shinbrot, National Advisory Committee for Aeronautics, Moffett Field, California*

Show that the Fermat equation $x^n+y^n=z^n$ has no nontrivial solution in integers for $n>2$ if $z<2^{1/n}/(2^{1/n}-1)$.

E 1290. *Proposed by R. E. Shafer, University of California Radiation Laboratory, Livermore, California*

If $|n \tan^{-1} x| \leqq \pi/2$, show that

$$n \tan^{-1} x = \tan^{-1} \frac{\operatorname{Im}(1+ix)^n}{\operatorname{Re}(1+ix)^n},$$

where $\operatorname{Im} f(z)$ denotes the imaginary part of $f(z)$ and $\operatorname{Re} f(z)$ denotes the real part of $f(z)$.

SOLUTIONS

Discussion of an Approximate Trisection

E 1256 [1957, 197]. *Proposed by W. B. Andreasen, Lockheed Aircraft Corporation*

Discuss the error involved in the following approximate trisection of a circular arc AB. On chord AB locate C such that $BC=BA/3$ and D such that $CD=7AB/6$. With D as center and DC as radius describe an arc to cut arc AB in the approximate trisection point E.

Solution by D. C. B. Marsh, Colorado School of Mines. Establish coordinate axes with origin at the circle's center and axes directed so that chord AB is parallel to and above the positive x-axis. Choose the scale so that the radius is 3. Label the points as: $B(3\cos\theta, 3\sin\theta)$, $A(-3\cos\theta, 3\sin\theta)$, $C(\cos\theta, 3\sin\theta)$, and $D(8\cos\theta, 3\sin\theta)$. If we assign E the coordinates $(3\cos\phi, 3\sin\phi)$, then E on circle $D(C)$ implies

$$(3\cos\phi - 8\cos\theta)^2 + (3\sin\phi - 3\sin\theta)^2 = (7\cos\theta)^2,$$

which reduces to

$$3 + \cos^2\theta = 8\cos\theta\cos\phi + 3\sin\theta\sin\phi,$$

and may be solved for ϕ to give

$$\phi = \operatorname{Arctan}\left(\frac{3\tan\theta}{8}\right) \pm \operatorname{Arccos}\left(\frac{3+\cos^2\theta}{\sqrt{9+55\cos^2\theta}}\right),$$

where $90° \geqq \theta \geqq 0°$, $90° \geqq \phi \geqq -90°$; the plus sign is used for the approximate trisection of the minor arc AB and the minus sign for the major arc. The relative error, valid for any scaled axis, is

$$(3\phi - \theta - 180°)/(180° - 2\theta) \quad \text{and} \quad (180° - \theta + 3\phi)/(180° + 2\theta)$$

in the respective cases.

The method is exact (by setting error equal to zero) only in the case where AB is a diameter ($\theta = 0°$), and trivially for the minor arc when AB is of length zero ($\theta = 90°$). For AB decreasing in length, the relative error in the major arc's trisection increases without bounds, being less than 0.01 for $\theta < 10°$ (roughly), less than 0.10 up to 40°, and 0.25 near 60°. The method is quite good for the minor arc, however, there being a maximum relative error of only about 0.007 when θ is 25° (or the given arc is around 130°). Following is a rough table:

θ	0°	5°	10°	15°	20°	25°	30°	35°	40°
arc	180°	170°	160°	150°	140°	130°	120°	110°	100°
% error	0.00	0.29	0.50	0.62	0.69	0.71	0.69	0.65	0.59
45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
90°	80°	70°	60°	50°	40°	30°	20°	10°	0°
0.51	0.43	0.35	0.27	0.19	0.13	0.07	0.03	0.01	0.00

Also solved by Julian Braun and (partially) by C. S. Ogilvy. Late solution by D. A. Breault.

A Property of the Complete Quadrilateral

E 1257 [1957, 197]. *Proposed by N. A. Court, University of Oklahoma*

(1) The medial triangle of each of the four triangles formed by the sides of a complete quadrilateral (q) taken three at a time is homological to the diagonal trilateral of (q).

(2) The four axes of the four homologies coincide.

Solution by the proposer. Let $(q) = abcd$ be the given quadrilateral. Its pairs of opposite vertices are the points $A = bc$, $P = ad$; $B = ca$, $Q = bd$; $C = ab$, $R = cd$, and the points $D = (BQ, CR)$, $E = (CR, AP)$, $F = (AP, BQ)$ are the vertices of the diagonal trilateral DEF of (q).

The sides $B'C'$, $C'A'$, $A'B'$ of the medial triangle $A'B'C'$ of the triangle $abc = ABC$ pass, respectively, through the midpoints U, V, W of the cevians

AP, BQ, CR of ABC. Now the latter three lines are the diagonals of (q), whence the points U, V, W lie on the Newton line n of (q). Thus the Newton line n is the axis of perspectivity of the medial triangle $A'B'C'$ of abc and the diagonal trilateral DEF of (q).

Similar results may be established for triangles bcd, cda, dab. Both parts of the proposition are thus proved.

Also solved by Josef Langr (synthetically as above) and D. C. B. Marsh (analytically).

An Infinite Radical

E 1258 [1957, 197]. *Proposed by Aaron Herschfeld, Canisius College*

Prove that a necessary and sufficient condition for the rationality of

$$R = \sqrt[3]{a + \sqrt[3]{a + \cdots}},$$

where a is a positive integer, is that $a = N(N+1)(N+2)$, the product of three consecutive integers. In that case find R.

Solution by D. A. Freedman, McGill University. Define $R_1 = \sqrt[3]{a}$, $R_n = \sqrt[3]{a+R_{n-1}}$. Now $R_2 > R_1$, and $R_k^3 - R_{k-1}^3 = R_{k-1} - R_{k-2}$, so that by induction $\{R_n\}$ is monotone increasing. Moreover, $R_1 < 1 + \sqrt[3]{a}$, and $R_{k-1} < 1 + \sqrt[3]{a}$ implies that $R_k^3 < a + 1 + \sqrt[3]{a} < (1 + \sqrt[3]{a})^3$, so that by induction $\{R_n\}$ is bounded. It follows that $\{R_n\}$ converges to a limit R. But then $R^3 - R - a = 0$. If R is rational and a integral, then R is integral, and $a = (R-1)R(R+1)$, the product of three consecutive integers. Hence the condition is necessary. It is also sufficient, since $R = N+1$ satisfies the equation $R^3 - R - N(N+1)(N+2) = 0$, and, as it is the only real root, it is the value of the radical.

Also solved by Robert Bart, Julian Braun and J. R. Holdsworth (jointly), R. E. Briney and D. A. Trumpler (jointly), C. N. Campopiano, Germain Casal and Ruben Perelis (jointly), W. V. Gamzon, A. M. Glicksman, Michael Goldberg, Bernard Greenspan, Cornelius Groenewoud, Emil Grosswald, H. J. Hauer, J. M. Howell, A. R. Hyde, I. M. Isaacs, Sidney Kravitz, Joseph Lewittes and Marshall Luban (jointly), Shen Lin, Joe Lipman, Wallace Manheimer, D. C. B. Marsh, J. B. Muskat, C. S. Ogilvy, Donald Passman, C. F. Pinzka, Montfort Plebstnoch, L. A. Ringenberg, T. A. Porsching, Michael Rosen, Azriel Rosenfeld, P. T. Schaefer, R. E. Shafer, Lawrence Shepp, D. D. Strebe, Chih-yi Wang, L. K. Williams, and the proposer. Late solution by D. A. Breault.

Several solvers generalized the problem by showing that a necessary and sufficient condition for the rationality of

$$R = \sqrt[n]{a + \sqrt[n]{a + \cdots}},$$

where a is a positive integer, is that $a = N(N^{n-1} - 1)$.

Ogilvy pointed out that this problem was essentially presented by C. W. Trigg in his solution of E 874 [1950, 186].

For the convergence of the sequence $\{R_n\}$, and of allied sequences, see Aaron Herschfeld, *On infinite radicals*, this MONTHLY [1935, 419–429].

Marsh and the proposer each pointed out that Cardan's formula yields the explicit expression

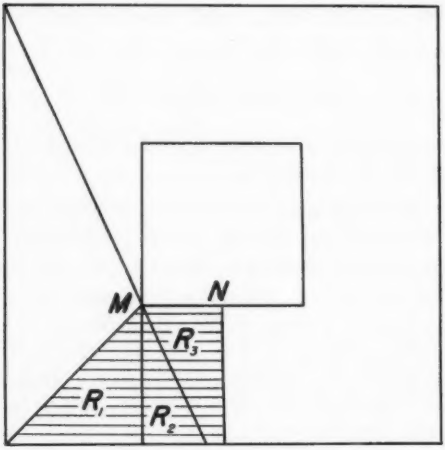
$$R = [a/2 + \sqrt{a^2/4 - 1/27}]^{1/3} + [a/2 - \sqrt{a^2/4 - 1/27}]^{1/3}.$$

The Chance that Two People Can See Each Other

E 1259 [1957, 197]. *Proposed by C. D. Olds, San Jose State College*

A building a feet square has a walk $b<a$ feet wide around it. Two persons are on the walk; find the chance that they can see each other.

Solution by A. R. Hyde, West Hartford, Connecticut, and D. C. B. Marsh, Colorado School of Mines. The figure shows a typical region R of the walk in which a person may be standing. Let convenient coordinate axes be taken.



[Hyde took origin at N and positive x-axis along NM; Marsh took origin at M and positive x-axis along MN.] Then at each point (x, y) of R there is a certain area, say $f(x, y)$, of the surrounding walk which is visible. The probability that a second person on the walk can be seen by the first person is then given by

$$p = \left[\int\int_R f(x, y)dxdy\right] \Big/ TR,$$

where T is the total area of the walk and R represents the area of region R.

Since $f(x, y)$ has discontinuities, met when the first person's position passes from one of the subregions R_1, R_2, R_3 into another, we calculate p by

$$p = \left[\sum_{i=1}^{3}\int\int_{R_i} f(x, y)dxdy\right] \Big/ TR.$$

A straightforward but tedious calculation yields

$$p = [(2 + a/b)^2 + \ln (1 + a/b)]/4(1 + a/b)^2. \tag{1}$$

The above calculation of p holds for $b \leqq a$. It may be shown [as did Hyde] that for $b > a$ two cases arise, according as $b \leqq$ or $> (1+\sqrt{5})a/2$. A figure shows that in the first of these cases R must be divided into four subregions, and in the second case into five subregions. Nevertheless, in both of these cases one again obtains the result (1). Hence (1) gives the desired probability for all relations of a to b. The limits of p are $1/4$ as $b/a \to 0$ and 1 as $b/a \to \infty$, giving two obvious checks on the formula for p. At $b = a$, $p = (9+\ln 2)/16 = 0.60582$, approximately.

Also solved by Julian Braun.

An Improper Integral

E 1260 [1957, 197]. *Proposed by Viktors Linis, University of Ottawa*

Evaluate $I = \int_0^{\pi/2} \cot\theta \ln \sec\theta d\theta$.

Solution by Emil Grosswald, University of Pennsylvania. The improper integral exists, as the integrand approaches zero at both limits of integration and is continuous over the open interval. Setting $x = \cos\theta$,

$$I = -\int_0^1 [(x \ln x)/(1-x^2)]dx = -\int_0^1 \left(\sum_{n=0}^{\infty} x^{2n+1}\right) \ln x dx.$$

Inversion of summation and integration is easily justified by the uniformity of convergence, so that

$$I = -\sum_{n=0}^{\infty}\left(\int_0^1 x^{2n+1} \ln x dx\right) = \sum_{n=0}^{\infty} (2n+2)^{-2} = (1/4)\sum_{m=1}^{\infty} m^{-2} = \pi^2/24.$$

Also solved by C. A. Church, Jr., F. J. Duarte, C. B. Germain, Bernard Greenspan, Cornelius Groenewoud, G. A. Harris, Jr., A. R. Hyde, I. M. Isaacs, Walter James, Seymour Kass, M. S. Klamkin, Marshall Luban, T. G. McLaughlin, D. C. B. Marsh, G. B. Parrish, Donald Passman, George Richardson and Dale Woods (jointly), L. A. Ringenberg, D. A. Robinson, R. E. Shafer, Lawrence Shepp, Arnold Singer, M. B. Stewart, D. D. Strebe, Chih-yi Wang, David Zeitlin, and the proposer. Late solution by J. L. Alperin.

The integral $I = \int_1^{\infty}[(x \ln x)/(1-x^2)]dx$ may be broken up by partial fractions to yield

$$I = (1/2)\int_0^1 [(\ln x)/(1+x)]dx - (1/2)\int_0^1 [(\ln x)/(1-x)]dx.$$

These latter integrals may be evaluated by formulas 509 and 510 of Peirce's *A Short Table of Integrals*.

Other useful substitutions for the transformation of the given integral are $x = \cos^2\theta$ and $x = \ln \sec\theta$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4758. *Proposed by Leonard Carlitz, Duke University*

Let $x_1, \cdots, x_n$ denote the roots of

$$(x - a + \sqrt{1 - 2ax + x^2})^n + (x - a - \sqrt{1 - 2ax + x^2})^n = 2(a^2 - 1)^{n/2}.$$

Evaluate the power sums

$$\sigma_k = x_1^k + \cdots + x_n^k \qquad (0 \leqq k < n).$$

4759. *Proposed by T. K. Pan, University of Oklahoma*

The tangential component of the absolute curvature vector of a unit vector field along a curve on a surface in ordinary space is known as angular spread of the field along the curve and is a generalization of geodesic curvature of a curve. Derive formulas for angular spread generalizing those formulas for geodesic curvature due to Beltrami and Bonnet. (See, *e.g.*, Eisenhart, *Differential Geometry*, 1909, pp. 136, 183.)

4760. *Proposed by G. U. Brauer, University of Minnesota*

Let $f(x)$ be a real function such that $f(0) = 0$, $f(1) \neq 0$, $\lim_{n\to\infty} f(n) = 0$, where n takes on positive integral values. Construct a sequence of integers $\{a_n\}$, $a_n \to \infty$, and a compact set C such that $f(a_n x) \to 0$ nonuniformly for x in C.

4761. *Proposed by Alfredo Jones, Institute of Mathematics and Statistics, Montevideo, Uruguay*

In a ring with identity and with proper ideals, there always exist maximal ideals. Is the statement true for rings with a nontrivial multiplication and with no identity?

4762. *Proposed by Samuel Beatty, University of Toronto*

Define γ_i by the relations

$$\sum_{r=1}^{n} \frac{\log^i r}{r} = \int_1^n \frac{\log^i x}{x}\, dx + \gamma_i + o(1).$$

Show that

$$\sum_{r=1}^{\infty}(-1)^{r+1}\frac{\log^p r}{r}=\frac{\lambda^{p+1}}{p+1}-\left[\lambda^p\gamma+\binom{p}{1}\lambda^{p-1}\gamma_1+\cdots+\binom{p}{p-1}\lambda\gamma_{p-1}\right],$$

($p=2, 3, \cdots$), where $\lambda=\log 2$ and γ is Euler's constant. This is a generalization of no. 4592 [1954, 350].

SOLUTIONS

Prime Numbers

4711 [1956, 669]. *Proposed by Morgan Ward, California Institute of Technology*

If $\pi(n)$ is the number of prime numbers not greater than n, show that

$$\pi(n)<\frac{5}{4}\,\frac{n}{\log n}$$

for every composite number n.

Remarks by J. B. Rosser, Cornell University. Lowell Schoenfeld called this inequality to my attention a couple of years ago. I don't know any elegant proof, but one can get a quite direct and straightforward proof by using the results and methods of my paper *Explicit bounds for some functions of prime numbers*, (Amer. J. Math., vol. 63, 1941, pp. 211–232.) By Theorem 25, p. 212, one concludes that the inequality holds for $e^{10}\leqq n\leqq e^{100}$. Finally, by a modification of the method explained on p. 217, one needs only a fairly short computation to verify that it holds for $1<n\leqq e^{10}$, except for $n=113$ (which is not composite.)

Similar results are in a paper by Schoenfeld and Rosser which should appear in the near future.

Related Inequalities

4712 [1956, 669]. *Proposed by J. V. Whittaker, University of California, Los Angeles*

Show that if $x_i\geqq 0$ $(i=1, \cdots, n)$ and $\sum_{i=0}^{n}1/(1+x_i)\leqq 1$, then $\sum_{i=1}^{n}2^{-x_i}\leqq 1$.

Solution by B. L. Foster and R. R. Phelps, University of Washington, Seattle. The result is obvious for $n=1$; while for $n=2$, $x_1, x_2\geqq 0$ and $(1+x_1)^{-1}+(1+x_2)^{-1}\leqq 1$ imply $x_1x_2\geqq 1$ and $x_1, x_2>0$. But then $2^{-x_1}+2^{-x_2}\leqq 2^{-x_1}+2^{-1/x_1}$, and the latter is no greater than 1 by problem E1063 [1953, 714].

The proof proceeds by induction. $\sum_{i=1}^{n}(1+x_i)^{-1}\leqq 1$ implies that

$$(1+z)^{-1}+\sum_{i=3}^{n}(1+x_i)^{-1}\leqq 1,$$

where z is defined by $(1+z)^{-1}=(1+x_1)^{-1}+(1+x_2)^{-1}\leqq 1$, and hence $z\geqq 0$. By the induction hypothesis,

$$2^{-z}+\sum_{i=3}^{n} 2^{-x_i} \leqq 1.$$

It remains only to show that $2^{-x_1}+2^{-x_2}\leqq 2^{-z}$ or $2^{-y_1}+2^{-y_2}\leqq 1$, where $y_i=x_i-z$ $(i=1, 2)$. This, however, follows from the case for $n=2$ upon verifying that $y_1, y_2\geqq 0$ and $(1+y_1)^{-1}+(1+y_2)^{-1}\leqq 1$.

Also solved by E. F. Beckenbach and J. W. Green, Peter Henrici, A. R. Hyde, Blagovest Sendov, Lawrence Shepp, Chih-yi Wang, L. E. Ward, Jr., and the proposer. Late solution by Robert Breusch.

Number Systems in Which Squares End in Square Digits

4713 [1956, 729]. *Proposed by Oystein Ore, Yale University*

In the number system with base $n=12$, every square number ends in one of the digits 0, 1, 4, 9, that is, in a square digit. Find all n for which this is the case.

Solution by F. D. Parker, Clarkson College of Technology, Potsdam, N. Y. The numbers 2, 3, 4, 5, 8, 12, 16 have the desired property, and there are no others. The proof runs in four parts, according as n is of one of the forms, $4m+1$, $4m+3$, $4m+2$, $4m$.

Let the base be of the form $n=4m+1$. Then

$$(2m)^2 = (m-1)(4m+1)+3m+1,$$
$$(2m-1)^2 = (m-2)(4m+1)+3m+3.$$

These remainders are the terminal digits if the squares are written in the system having base n unless $3m+3\geqq n$, in which case we have $(2m-1)^2=(m-1)\cdot(4m+1)+2-m$. Since two squares cannot differ by 2 there are no solutions except in this special case, $m\leqq 2$. But $3m+1$ and $2-m$ are both squares only for $m=1$, whence $n=5$.

Suppose $n=4m+3$. Then

$$(2m)^2 = (m-1)(4m+3)+m+3, \quad (2m+1)^2 = m(4m+3)+m+1.$$

Since they differ by 2, both remainders cannot be squares except when $m=0$ and $n=3$.

If $n=4m+2$, then

$$(2m)^2 = (m-1)(4m+2)+2m+2, \quad (2m+1)^2 = m(4m+2)+2m+1.$$

These remainders cannot both be squares unless $m=0$ and $n=2$.

Finally, let the base be $n=4m$, and let $m^2=K\cdot 4m+r$, $0\leqq r<4m$. Then $(m+1)^2=K\cdot 4m+r+2m+1$,

$$(m+2)^2 = (K+1)4m+r+4, \quad (m+3)^2 = (k+1)4m+r+2m+9.$$

Since, from the first equation, r must be divisible by m, it is either 0, m, $2m$ or $3m$. If $m>4$, the requirement that the remainders for m^2 and $(m+2)^2$ be squares demands $r=0$ but then, in order that the remainders for $(m+1)^2$ and $(m+3)^2$ be squares, m must also be 0, an impossibility since $n>1$. Inspection of the cases $m=1, 2, 3, 4$ completes the proof with the result stated at the outset.

Also solved by H. F. Bennett, W. J. Blundon, D. A. Breault, Robert Breusch, N. J. Fine, A. S. Gregory, A. R. Hyde, and D. C. B. Marsh. In several of these solutions the proof that there are no other values of n was incomplete.

Editorial Note. The present result follows as an easy consequence from that of problem no. 4737 [1957, 277] for which a solution will appear in approximately four months.

Extrema of a Polynomial

4714 [1956, 729]. *Proposed by Chandler Davis, Columbia University and the New School for Social Research*

For a real polynomial $P(x)$ of degree n, denote the zeros of $P'(x)$ (multiplicity counted) by $\xi_i, i=1, \cdots, n-1$. Assume all ξ_i real, and let $\xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_{n-1}$. Now to what extent are the numbers $P(\xi_i)$ arbitrary? More precisely, give necessary and sufficient conditions on an $(n-1)$-tuple of real numbers $\eta_1, \cdots, \eta_{n-1}$, in order that there exist a polynomial P such that $P'(\xi_i)=0$, $P(\xi_i)=\eta_i$, $i=1, \cdots, n-1$.

Solution by the proposer. The numbers $(-1)^i(\eta_i-\eta_{i-1})$, $i=2, 3, \cdots, n-1$, must be either all nonnegative or all nonpositive. This condition is obviously necessary. The proof of sufficiency follows, assuming arbitrary $(-1)^i(\eta_i-\eta_{i-1}) \leqq 0$, $\eta_1 \geqq 0$.

It is no restriction to consider only the case $0 \leqq \xi_1 \leqq \xi_{n-1} \leqq 1$; define $\xi_0=0$, $\xi_n=1$. Only polynomials P such that $P(0)=0$ will be considered. It will be shown that the numbers $\phi_i=(-1)^{i-1}\{P(\xi_i)-P(\xi_{i-1})\}$, $i=1, \cdots, n$, can be chosen arbitrarily provided only $\phi_i \geqq 0$. (The trivial case $\phi_i \equiv 0$, $P(x) \equiv 0$ will also be excluded.) Using the notation

$$P(x) = A\int_0^x \prod_{i=1}^{n-1} (\xi_i - x)dx, \qquad A > 0,$$

justified by the assumptions made above, define for convenience $\delta_i=A(\xi_i-\xi_{i-1})$, $i=1, \cdots, n$. Now the main point in the proof is to consider the ϕ_i as n functions of the n variables $\delta_1, \cdots, \delta_n$.

First, $(\delta_1, \cdots, \delta_n)$ ranges over the set $\mathcal{P}^n$ of n-space $\mathcal{E}^n$ defined by the requirement that all coordinates be nonnegative and not all be zero; $(\phi_1, \cdots, \phi_n)$ is in $\mathcal{P}^n$ also, it must be shown that it can have any value in $\mathcal{P}^n$. Now the boundary is mapped into the boundary and the interior into the interior, for $\delta_i=0$ if and only if $\phi_i=0$. Every ray through the origin is mapped onto some ray through the origin (two points on the same ray differ only in the values of A). Evidently $\xi_1, \cdots, \xi_{n-1}, A$ are continuous functions of the δ_i, and the ϕ_i are continuous functions of $\xi_1, \cdots, \xi_{n-1}, A$. The associated Jacobians will be

proved nonzero for $(\delta_1, \cdots, \delta_n)$ in the interior of $\mathcal{P}^n$, that is, for $\xi_i > \xi_{i-1}$, $i=1, \cdots, n$; this will give the result.

It is easy to compute $\partial(\xi_1, \cdots, \xi_{n-1}, A)/\partial(\delta_1, \cdots, \delta_n) = A^{-n+1}$. For $i=1, \cdots, n$, define

$$a_{ij} = \frac{\partial \phi_i}{\phi \xi_j} = (-1)^{i-1} \int_{\xi_{i-1}}^{\xi_i} \frac{P'(x)}{\xi_j - x}\, dx, \quad j = 1, \cdots, n-1,$$

$$a_{in} = \frac{\partial \phi_i}{\partial A} = \frac{(-1)^{i-1}}{A} \int_{\xi_{i-1}}^{\xi_i} P'(x)dx.$$

It must be shown that the determinant of the matrix $\|a_{ij}\|$ is not zero.

If it were zero, there should be some linear relation of the form $\sum_1^n c_j a_{ij} = 0$, holding independent of i. This would mean that the function

$$F(x) = \sum_{j=1}^{n-1} c_j \frac{P'(x)}{\xi_j - x} + \frac{c_n}{A} P'(x),$$

a polynomial of degree $n-1$, would integrate to zero between $x=\xi_{i-1}$ and $x=\xi_i$, $i=1, \cdots, n$; hence that $\int_0^x F(x)dx$, a polynomial of degree n, would be zero at the $n+1$ points $\xi_0, \cdots, \xi_n$, so that $F(x) \equiv 0$. But the n summands of $F(x)$ are easily seen to be linearly independent, so the only possibility is all $c_j = 0$. This completes the proof.

Remark. The argument shows that $\phi_1, \cdots, \phi_n$ determine P uniquely provided no ϕ_i is zero. This restriction can be removed by a refinement of the same argument. It would be interesting to find an "interpolation" formula for P in terms of the ϕ_i.

Editorial Note. A. W. Goodman asks the analogous question for the complex domain: Can the branch points of (the image domain of) a polynomial be prescribed in advance? That is, given $n-1$ complex numbers $B_1, \cdots, B_{n-1}$, are there a polynomial $P(x)$ of degree n, and $n-1$ complex numbers $C_1, \cdots, C_{n-1}$ such that $P'(C_k) = 0$, $P(C_k) = B_k$, $k=1, \cdots, n-1$? One may assume the B_k are all distinct for simplicity.

Simultaneous Equations

4715 [1956, 729]. *Proposed by R. S. Underwood, Texas Technological College*

Find real solutions of the equations

$$x^2 + y^2 + z^2 + u + v = D, \qquad x^2 + \frac{y^2}{2} + \frac{z^2}{3} + \frac{u^2}{4} + \frac{v^2}{5} = 1,$$

in case (a) $D=-3$, and (b) $D=15/4$. What can be said about the solutions for other values of D?

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* Subtracting the first from three times the second gives

$$2x^2+\frac{1}{2}y^2+\frac{3}{4}\left(u-\frac{2}{3}\right)^2+\frac{3}{5}\left(v-\frac{5}{6}\right)^2=\frac{15}{4}-D,$$

so that D is not more than $15/4$. If $D=15/4$, then every term on the left vanishes, giving the solutions $(x, y, z, u, v)=(0, 0, \pm 3/2, 2/3, 5/6)$.

Adding the first to $3/2$ times the second gives

$$\frac{5}{2}x^2+\frac{7}{6}y^2+\frac{3}{2}z^2+\frac{3}{8}\left(u+\frac{4}{3}\right)^2+\frac{3}{10}\left(v+\frac{5}{3}\right)^2=D+3.$$

Thus D cannot be less than -3. For $D=-3$, the unique solution is $(0, 0, 0, -4/3, -5/3)$. There are no real solutions for values of D outside the range $-3\leqq D\leqq 15/4$.

II. *Solution by N. J. Fine, University of Pennsylvania.* Let $\alpha=D-r^2=D-(x^2+y^2+z^2)$, $\beta=1-(x^2+y^2/2+z^2/3)$. Then the system $u+v=\alpha$, $u^2/4+v^2/5=\beta$ has a real solution if and only if $\alpha^2\leqq 9\beta$; the solution is unique when equality holds. Therefore D must satisfy

$$r^2-3\sqrt{\beta}\leqq D\leqq r^2+3\sqrt{\beta}. \tag{1}$$

For all points of the ellipsoid E, defined by $\beta\geqq 0$, we have $r^2\geqq 0$, $\sqrt{\beta}\leqq 1$, so $r^2-3\sqrt{\beta}\geqq -3$, and equality holds only for $(x, y, z)=(0, 0, 0)$. Thus for $D=-3$, there is the unique solution $(0, 0, 0, -4/3, -5/3)$. To determine the largest possible value of D, we maximize $F=r^2+3\sqrt{\beta}$ over E. On the boundary of E, $F=r^2\leqq 3$. In the interior, the partial derivatives of F must vanish, so (i) $x=0$, (ii) $y=0$ or $\beta=9/16$, (iii) $z=0$ or $\beta=1/4$. Testing the alternatives, we find the maximum of F to be $15/4$, attained for $x=y=0$, $z=\pm 3/2$. Thus, for $D=15/4$, we have two solutions $(0, 0, \pm 3/2, 2/3, 5/6)$. For all other D, $-3<D<15/4$, there are infinitely many points of E for which $r^2-3\sqrt{\beta}=D$ or $r^2+3\sqrt{\beta}=D$. For each of these, (1) is obviously satisfied and a solution exists.

Also solved by G. E. Bredon, Emil Grosswald, Edgar Karst, D. C. B. Marsh, R. C. Read, Blagovest Sendov, and the proposer.

Editorial Note. Marsh adds the comment, "For those who are adept at employing the proposer's methods of extended analytic geometry these same results are immediately found by way of the simultaneous solutions of a filled ellipse and parabola." See Underwood, *Extended analytic geometry as applied to simultaneous equations*, this MONTHLY, 1954, pp. 525–542.

An Equation of Motion

4716 [1956, 729]. *Proposed by M. S. Klamkin, AVCO Research and Development, Lawrence, Mass.*

Determine the equation of motion if $\overline{V}_s=\lambda\overline{V}_t$, where $\overline{V}_s$ and $\overline{V}_t$ are the averages of velocity with respect to distance and time, respectively, in any time interval starting at $t=0$. What is the minimum eigenvalue λ?

Solution by N. J. Fine, University of Pennsylvania. Assuming that $s(0)=0$, we see that $\overline{V}_t=s/t$, and so

$$\frac{1}{s}\int_0^s v\,ds = \lambda\frac{s}{t}.$$

Multiply by s and differentiate, to get

$$v = \lambda\left(2\frac{s}{t} - \frac{1}{v}\frac{s^2}{t^2}\right) \quad \text{and} \quad v = \mu\frac{s}{t},$$

where $\mu=\lambda\pm\sqrt{\lambda^2-\lambda}$. Hence $s=ct^\mu$. Except for the trivial case $\mu=\lambda=0$, we must have $\mu>1/2$ to ensure finiteness of $\overline{V}_s$, and $\lambda=\mu^2/(2\mu-1)\geqq 1$. Therefore every $\lambda\geqq 1$ is an eigenvalue, with the solution $s=ct^\mu$, $\mu=\lambda\pm\sqrt{\lambda^2-\lambda}$.

Also solved by Mary Payne, Chih-yi Wang, and the proposer.

Fixed Points of Entire Functions

4717 [1956, 729]. *Proposed by Patrick Gallagher, Harvard University*

Let $f(z)$ and $g(z)$ be entire functions, each without fixed points, not both linear. Then $f(g(z))$ has infinitely many fixed points.

Solution by Peter Henrici, University of California, Los Angeles. Since the entire function $f(z)$ has no fixed points, the function $f(z)-z$ is an entire function without zeros and hence can be represented in the form $e^{F(z)}$, where $F(z)$ is again entire. Similarly for $g(z)$ in terms of a certain entire function $G(z)$:

$$f(z) = e^{F(z)} + z, \qquad g(z) = e^{G(z)} + z.$$

A point z is a fixed point of $f(g(z))$ if and only if it satisfies the equation $e^{F(e^{G(z)}+z)} = -e^{G(z)} = e^{G(z)+i\pi}$. For this to occur it is necessary and sufficient that, for some integer n, $F(e^{G(z)}+z)=G(z)+(2n+1)i\pi$. Since not both of the functions $f(z)$ and $g(z)$ are linear, at most one of the functions $F(z)$ and $G(z)$ is a constant. It follows that $H(z)=F(e^{G(z)}+z)-G(z)$ is a nonconstant entire function and therefore, by Picard's theorem, assumes every value with at most one exception. Hence it assumes infinitely many of the values $(2n+1)i\pi$, *i.e.*, there are infinitely many fixed points.

Also solved by Joshua Barlaz, Robert Breusch, D. S. Carter and G. M. Wing, D. J. Newman, T. J. Rivlin, Blagovest Sendov, John Wermer, L. K. Williams, and the proposer.

Editorial Note. Rivlin points out that this and related results are contained in P. C. Rosenbloom, *The Fix-Points of Entire Functions* (Comm. Sém. Math. Univ. Lund, Tome Supplémentaire, 1952, dedié à Marcel Riesz.)

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

NOTICE

It has been suggested that this department prepare a "list of mathematical books suitable for small college libraries." If there is sufficient interest, such a list will be prepared. If *you* have any nominations for books especially suitable for small college libraries, please send their titles to Richard V. Andree, University of Oklahoma, Norman, Oklahoma.

The Mathematics of Investment. By Roger Osborn. Harper, New York, 1957. viii+162 pp. +117 pp. of tables. $4.25.

The main difference between this text and the myriad of others on the same subject seems to be that the present work contains not much more material than can be covered in a one-semester, three-hour course. The result is that bonds are given summary treatment and life insurance is not discussed at all. This reviewer looks upon the omission of insurance as a fatal defect in the book, and as one that will make the text unsuitable for most classes in the mathematics of finance. The author also appears to be convinced that students will be required to memorize the plethora of formulae, and thus he devotes attention to developing memorizable ones. In this connection the footnote beginning on page 82 is amusing. For therein the author develops the easily computed

$$S = R(s_{\overline{n+1}|i} - 1)$$

formula for the sum of an annuity due, but rejects it summarily, partly on the grounds that it is difficult to memorize.

On the credit side, the book contains one of the clearest statements known to the reviewer of the distinction between an annuity and an annuity due, as well as carefully explained illustrative exercises and diagrams. The text is readable and the printing job satisfactory.

If one is seeking a text for a course which does not cover life annuities or insurance, and if one desires a book which places a premium on memorization (see the footnote on page 117 for the author's philosophy on this point, and for an unfortunate reflection on the character of students in general), then this text is quite suitable.

R. L. SAN SOUCIE
University of Oregon and Sylvania Electric Co.

Fun with Mathematics. By Jerome S. Meyer. The World Publishing Co., Cleveland and New York, 1952. x+176 pp. $2.75.

Instructors of secondary school or junior college mathematics who are looking for material to interest their promising students will find a variety of useful (although ungraded) material in this book, dedicated "to all young people who love mathematics." Little background is needed, although plane geometry and elementary algebra will be helpful, and the chapter on π, i, e, and logarithms requires the binomial expansion and trigonometry.

The book is readable and inviting, with large type, many charts and illustrations, and chapter headings that pique curiosity: "Babe Ruth Hit 111100 Home Runs" (binary system), "Curves That Control Our Lives" (conic sections). Other topics include number oddities, Fibonacci numbers, magic squares, trigonometry, the limiting process, fallacies and problems, and mathematical "how-to-do's" that will interest the budding engineer. Much of the material is given without explanation, although occasionally hints are offered; on the other hand, the section on the limiting process contains a lengthy and satisfactory exposition on an intuitive basis. Instructions for constructing a sun dial, several specific nomographs, and a slide rule are directive only; the reviewer thought it odd, since a previous chapter dealt with logarithms, that the slide rule directions contained no reference to them, actual or implied.

In general the mathematical presentation is satisfactory, although in a section devoted to fallacies one finds "$\sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1}$ (which is correct)," followed by "$\sqrt{1}$ is either $+1$ or -1," to explain the resulting fallacy, and there are a few statements that could be more precise ("there can be *thousands* of differently shaped ellipses $\cdots$," not long after a section devoted to an explanation of the concept of infinity). It might be added that, while the book can be put into the hands of an individual student, it would be well if there were a well-trained mentor in the background to give direction, and to answer some inevitable questions. Also, it should be remembered that the primary purpose of the book is to stimulate and give constructive amusement, rather than to give formal instruction.

BESS E. ALLEN
Wayne State University

Nonparametric Methods in Statistics. By D. A. S. Fraser, Wiley, New York, 1957. x+299 pp. $8.50.

In *Nonparametric Methods in Statistics* the author, D. A. S. Fraser, has successfully attempted to collect and unify the different developments of nonparametric methods in a book intended as a second course in mathematical statistics. The text presupposes a knowledge of the calculus and familiarity with an introduction to statistics such as found in Hoel's *Introduction to Mathematical Statistics* or Mood's *Introduction to the Theory of Statistics*. The essential ideas of measure theory are introduced and illustrated as needed. Naturally, the stu-

dent who is more familiar with measure theory will find the reading ease improved. The first two chapters consisting of 124 pages treating sample space (measure space, measures and probability measure, expectation and conditional probabilities, sufficient statistics, completeness) and statistical inference (decision problem, estimation of real parameters, hypothesis testing, confidence and tolerance regions) have been used for an undergraduate course surveying recent small sample methods. The remainder of the book is intended for a graduate course on the applications of these methods in the nonparametric branch of statistics. This portion of the text, 172 pages, contains five chapters on nonparametric problems (single sample, randomness, randomized blocks, and more general designs), estimation of real parameters and tolerance regions, theory of hypothesis testing (unbiasedness, most powerful tests, most powerful rank tests, likelihood ratio method), limiting distributions (general theorems concerning limiting distributions, central limit theorems, limiting distribution of **U** statistics, Wald-Wolfowitz limit theorem, limiting distribution or runs, and additive partition functions), and large sample properties of tests (consistency, criterion for the relative efficiency of tests, efficiency of some conditional tests, efficiency of a rank test). Excellent and extensive lists of problems for solution are supplied at the end of each chapter. The author uses these problems to develop portions of the theory as well as to illustrate, apply, and extend it. Fine reference and bibliography lists conclude each chapter. The text and references will impress the student with the recency of the developments of this subject.

JOHN C. BRIXEY
University of Oklahoma
Norman, Oklahoma

Introduction to Statistical Analysis. Second Edition. By Wilfrid J. Dixon and Frank J. Massey, Jr. McGraw-Hill, New York, 1957. xiii+488 pp. $6.00.

This textbook, designed for a one-year basic course in statistics, follows the recommendations of the "committee on teaching of statistics of the National Research Council." Both editions of the book share these characteristics: illustrative examples have been taken from a wide variety of fields such as agriculture, engineering, and medical research; the problem lists are well organized and comprehensive; students are encouraged to carry out simple sampling experiments; discussions are concise; the chapters are unusually self-contained; calculus is not used; in general, formulas are stated without proofs but with some indication of their limitations; and wide use is made of tables contained in the back of the book.

The major changes in the Second Edition are: a chapter on probability; the chapters on statistical inference and analysis of variance have been largely rewritten; almost all problem lists have been augmented; the references have been combined into one extensive list at the back of the book; the old tables have been

expanded and 7 new tables added; and the revolting caricatures have been omitted. In 29 pages the chapter on probability covers the addition and multiplication theorems; the binomial, hypergeometric, and Poisson distributions; and closes with a brief mention of continuous chance variables. Most of the problems deal with sampling situations both with and without replacement.

On the whole the book is carefully written. However, two minor complaints come to mind. On page 30 the authors introduce sampling by stating that "any subset of a population is a sample from that population." On page 36 and again on page 42 the samples enumerated in the illustrative example are *ordered* subsets. It seems to this reviewer that the troublesome question of the relevance of order in sampling might have received more attention. In testing hypotheses set up for possible rejection, this book uses the two alternatives, "reject" and "accept." On pedagogical grounds this reviewer prefers the alternatives, "reject" and either "not reject" or "reserve judgment." The type is large and clear in both the text material and the tables. An interesting misprint "petmutations" occurs on page 356.

Anyone in the market for a statistics text with wide coverage below the calculus level should examine this book.

V. V. LATSHAW
Lehigh University

Theory of Approximations. By N. I. Achieser (translated from the Russian by Charles J. Hyman). Frederick Ungar, New York, 1956. 307 pp. $8.50.

The reader is referred to a review of the original Russian edition of this book by A. Zygmund in *Mathematical Reviews*, vol. 10, 1949, p. 33, for an excellent discussion of its contents. Its translation into English makes available to many readers a very systematic and complete development of the theory of approximation, not previously available in one volume. Not only is the treatment of the subject complete, but those basic concepts germane to the subject are developed in brief but sufficient detail, and the book may serve as a very good reference work in which can be found concise definitions and theorems relating to many topics in the field of analysis. Among those topics particularly well treated are those of a metric space, linear normalized space, Hilbert space, separable and complete spaces, orthonormal systems, linear functionals, Fourier series and transforms, multimonotonic functions, and functionals of exponential type.

The book is extremely well arranged with sufficient groundwork laid in the first chapter of fifty pages for the treatment of special types of approximation given in the remainder of the book. Here for example is given a proof of the fundamental theorem of approximation theory in linear normalized spaces. The many examples of applications of the theory to Hilbert space as well as the geometric interpretations given contribute a great deal to the clarity of the treatment.

In Chapter II, Chebyshev's theorem on the approximation of a function

$f(x)$ in terms of a given function $s(x)$ and a fractional expression involving two polynomials in x is proved. Several examples of applications of the theorem are given, one of considerable interest involving elliptic functions. The treatment is then specialized to consideration of the best trigonometric sum approximation to a periodic function and other particular problems.

Chapter III on harmonic analysis treats Fourier series briefly and contains a proof of the convergence of a Fourier series of an integrable function of bounded variation. Fourier transforms and other integral operators are studied. Although these topics have an interest in themselves, they are introduced because of their application to the theory of approximation and convergence. Chapter IV takes up the special topic of extremal properties of integral transcendental functions of exponential type. The author is thus led to two generalizations of Bernstein's inequality. In Chapter V the relation between rapidity of convergence of the approximating functions to the differentiability properties of the given function is treated and the theorems of Bernstein and Jackson on this subject are proved and generalized. Wiener's treatment of approximation to a given function by functions of very general type is considered in Chapter V.

Pages 243 to 295 are used for statements and solutions of some interesting illustrative extremal and approximation problems. For those who wish to try these problems it should be noted that the author classifies them as elementary. The table of contents reflects the orderly development followed in the text. Notes, references, and the table of contents are also adequate.

It must be said that this treatise on approximation theory is well planned and written in such a way that a graduate student well grounded in the usual graduate courses in analysis or the nonspecialist can study it with profit and pleasure. The translator should be congratulated on the smoothness and readability as well as the accuracy of the English text. The professional mathematician should find the organization of the work as well as the brevity and clarity of the proofs pleasing. The book would be suitable as a reference work for a graduate course or seminar in what is certainly a field of analysis of broad application.

O. H. HAMILTON
Oklahoma State University

Proceedings of the International Symposium on Algebraic Number Theory. The Science Council of Japan. Pan-Pacific Press, 1957. xx+267 pp. $5.

This is a collection of research papers on algebraic number theory and allied topics, delivered at an International Symposium in Tokyo and Nikko, Japan, on September 8–13, 1955. There are twenty papers, of which approximately half were contributed by mathematicians from France, Germany, India, and the United States, and the remainder, along with seventeen short notes, by Japanese mathematicians. The subject matter ranges widely over algebraic number theory, such as the idéle-class group of an algebraic number field, Siegel's

modular functions, the generalized principal ideal theorem, and the cohomology of algebraic number fields. There are papers on algebraic geometry, especially on algebraic and abelian varieties, and several papers touching to some degree on number theory. A special emphasis was laid on possible extensions of the class field theory, and several papers deal with a generalization of the theory of complex multiplication. A detailed review paper by paper is not appropriate here, but to give some idea of the outstanding group of mathematicians participating in the Symposium and contributing to this volume, we list the nineteen authors of the major papers in the order of their appearance: A. Weil, G. Shimura, Y. Taniyama, M. Deuring, E. Artin, R. Brauer, K. Iwasawa, T. Tannaka, T. Nakayama, T. Kubota, K. Yamazaki, K. G. Ramanathan, I. Satake, C. Chevalley, A. Néron, Y. Nakai, J.-P. Serre, M. Nagata, and D. Zelinsky.

IVAN NIVEN
University of Oregon

Introduction to Finite Mathematics. By John G. Kemeny, J. Laurie Snell and Gerald L. Thompson, Prentice-Hall, New Jersey, 1957. 372+xi pp., $5.00.

In the growing tide of new elementary texts, this one stands out as the most radical, and to my mind happy departure from tradition. No warming over of high school algebra or gentle calculus background this volume, but a substantial dose of modern topics based upon *finite* sets. The five core chapters are: compound statements, sets and subsets, partitions and counting, probability theory, and vectors and matrices. Careful organization and efficient use of concepts is obvious throughout, giving the exposition a quality reminiscent of more advanced texts. The transition from this book to later courses should be smoother than usual, but the transition to the book itself is another matter.

The authors, although careful to include many examples and problems, may have gone too far in the direction of sophistication: the style, precise and simple, is spare. For example, there are unembellished definitions such as "A *column vector* is an ordered collection of numbers written in a column."—the first sentence of Chapter 5. But, let it be recorded, the Dartmouth freshmen on which the text has been *successfully* tested were not specially selected. However, their instructors were the authors, and the lectures, no doubt, richly supplemented the text. Good lectures—always desirable—may well be necessary when using this book.

The book is also intended for those increasingly common behavioral science graduate students who sense a need to know some mathematics. For them there are two chapters of applications, covering aspects of linear programming and two-person game theory and nontrivial topics from sociometrics, genetics, learning theory, anthropology, and economics.

The main fault of commission is, I believe, the attempt to discuss without sufficiently clear delineation the notions of measure, relative frequency, and credibility of statements in the probability chapter. The book's main omission

(especially unfortunate for the behavioral scientists) is that collection of ideas, beginning with product sets, and including relations, orderings, functions, and axiomatic applications such as utility theory.

In sum, even though I have some specific reservations, I would commend this among beginning texts as an exciting and remarkably successful attempt to tap a different, and clearly important, lode of mathematics. Its influence should be widely felt, even though it may be deemed too mathematically mature for some freshmen courses.

DUNCAN LUCE
Harvard University

Mathematics of Finance. By Hugh E. Stelson. Van Nostrand, Princeton, N. J., 1957. xii+327 pp. $5.50.

This book differs from the usual mathematics of finance text in its treatment of both the nontechnical and the theoretical aspects of this subject. Its emphasis on the practical viewpoint may be indicated by its realistic discussion of "Consumer Loans," "Buying a Home," "The Nature of Life Insurance." These are the headings of three of the book's fifteen chapters. The basic ideas of these topics are presented simply and clearly. The book's attention to technical details and soundness of theory may be indicated by the unusually large number of footnotes. There are twenty footnotes to literature alone, and many others which provide explanatory comments or details for proofs.

The book is well-written in a concise style. The paragraphs are exceptionally short, yet the explanations are clear and sufficiently detailed. All the standard topics in the mathematics of finance are included and treated competently. There is an adequate number of realistic examples and exercises. Each chapter ends with a list of formulas and symbols. The tables are excellent. Eight of the ten tables included are reprinted from Glover's *Tables of Applied Mathematics.*

The preface and the introduction to Chapter 15, which is a review of "Preparatory Topics from Algebra," imply that the students using this text should be well-grounded in algebra. In addition, familiarity with limits is needed for the sections on perpetuities and continuously convertible interest. Also, a knowledge of calculus is essential for the chapter on "Continuous Annuities." However, the author indicates in the preface that all of these portions of the text, along with some others, "can be omitted for a short course with emphasis on applications."

On the assumption that college algebra is a minimum prerequisite for the course in mathematics of finance, the reviewer considers that Professor Stelson has accomplished his purpose of preparing an excellent text for students in mathematics or business administration.

H. S. KALTENBORN
Memphis State University

Nonparametric Statistics. By Sidney Siegel, McGraw-Hill, New York, 1957. xvii+312 pp. $6.50.

Most of the techniques of statistical inference presented in textbooks on statistics for social scientists are valid only under quite stringent assumptions on the nature of the underlying distributions, *e.g.*, that all observations were obtained from normally distributed populations. Since these assumptions are quite often not fulfilled for the available data, these "classical" statistical techniques may, if applied, lead to doubtful conclusions. To remedy this situation, statisticians have been developing a number of procedures which can be correctly applied under very mild restrictions on the class of populations considered. Most of these procedures presuppose only that the variates considered have continuous distributions, but require no further assumptions on the form of the distributions. These very general statistical methods, developed and published mostly in the last two decades, are known to statisticians as "nonparametric" or "distribution-free" methods of statistical inference.

The book by Siegel is written for the use of social scientists or, in the author's terminology, for behavioral scientists, and represents a pioneering effort to introduce the behavioral scientists to the use of distribution-free statistical techniques. The author states specifically that it is his intention to "make this book fully comprehensible to the reader who has had only introductory work in statistics . . . " and "whose mathematical training is limited to elementary algebra." This specific purpose of the book explains some peculiarities of style which, to a professional statistician may be somewhat disconcerting: the presentation is verbose and frequently contains sentences or paragraphs that contribute little to the progress of thought, and the exposition is repetitious to an extent that may at times confuse the reader. More serious than these qualities of style are some material shortcomings. Basic concepts are often presented in a manner which is likely to give the reader an incorrect picture. To mention some examples: from the book it would appear that the main, possibly the only, use of nonparametric statistics in behavioral sciences consists in testing hypotheses, while problems of estimation are mentioned exactly once in the introduction and distribution-free tolerance limits are not mentioned at all. The concept of power of a test is explained without sufficient emphasis on the fact that the power is defined only for a specified alternative, and the frequently occurring phrase "for these data the . . . test exhibits greater power to reject H_0 than the . . . test" adds to the confusion. The discussion of power-efficiency which follows the presentation of each test may suggest to the reader some vague idea of comparative goodness of tests, but it is doubtful that he will clearly understand what this means. The same mathematical models are presented in various sections of the book under different names, for example, the 2×2 table or the binomial test appear repeatedly under different labels; possibly the author wanted to show how the same technique can be applied to rather different types of research

problems, but then the reader would deserve a hint that this is the same procedure over and over again.

In spite of these shortcomings, *Nonparametric Statistics* is a useful and significant book. It is the first presentation in book form of a body of knowledge that has been accumulating in scattered articles in professional periodicals and which is of considerable importance to the practical researcher. The intuitive motivation of the techniques presented is generally plausible and should prove appealing and informative to the behavioral scientist, even if it does not meet the standards of a mathematically trained reader. Every technique presented is illustrated by examples couched in terms of concrete applications and every step is patiently explained. An appendix consisting of fifty-five pages of tables makes accessible for the first time a large amount of tabular material which previously could be found only in professional journals.

In summarizing, this reviewer feels that regardless of the reservations he has with regard to the accuracy of formulation and sometimes to the logic of presentation, Siegel's book is to be considered an important step in the process of introducing hitherto little known but powerful and useful statistical methods into the work of behavioral scientists.

Z. W. Birnbaum
University of Washington

BRIEF MENTION

Mathematics for Psychologists, Examples and Problems. By Robert R. Bush, Robert P. Abelson and Ray Hyman. Social Science Research Council, 1957. vi+86 pp. $2.00.

Mathematicians interested in providing mathematical training for social scientists will welcome this collection of problems. The fields of testing and measurement, psycho-physics, physiological psychology, and learning are each well represented in this book, and there are several examples of current research on small groups, sociometry and related areas of social psychology. In each case there are examples and references to published material. The book is divided into four general parts: applications to calculus 24 pages, mathematical foundations 10 pages, matrix algebra 22 pages, probability theory (including Markov chains) 20 pages. An appendix gives answers to the problems. A bibliography of one hundred and twenty four items concludes this brief but welcome volume.

System Engineering. By Harry H. Goode, and Robert E. Machol. McGraw-Hill, New York, 1957, xii+551 pp. $10.00.

This introduction to the design of large-scale system engineering will certainly find its way into the library of every university in which a large-scale computer is being considered or is in use.

Bibliography of Russian Mathematics Books. By George E. Forsythe. Chelsea, New York, 1957, 106 pp. $3.95.

The History of Mathematics. By Joseph E. Hofmann. Philosophical Library, New York, 1957. xi+132 pp. $4.75.

This brief books deals with what today might be called Ancient Mathematics, from about 2000 B.C. to about 1650 A.D.

A Short Dictionary of Mathematics. By C. H. McDowell. Philosophical Library, New York 1957. xiii+63 pp. $2.75.

This dictionary limits itself deliberately to high school mathematics of arithmetic, algebra, plane geometry and trigonometry. Some of the terms are not defined precisely as this reviewer would define them, even for this limited group. For example, "Absolute means pure, unmixed or exact." Other examples, "Area is any plane surface, the extent of the surface of any figure included within any three or more straight lines or within any closed line or lines," or "Divergent means receding further and further away." This reviewer would much prefer the carefully written James and James mathematical dictionary even for a high school library.

How to Solve It. By G. Pólya. Doubleday, Garden City, N. Y., 1957. 253 pp. $0.95.

The appearance of a low cost paperback edition of this well-known book will be welcomed by teachers and students everywhere.

Scientific German. By George E. Condoyannis. Wiley, New York, 1957. $2.50.
Scientific French. By William N. Locke. Wiley, New York, 1957. $2.25.

These two concise companion volumes fulfil a long-felt need of mature students who desire to acquire a reading (not speaking or writing) knowledge of scientific (not literary) German and French. Grammatical concepts such as the passive voice and impersonal verb, so important in reading scientific work, are not slighted. No childish "John saw the pencil" drill here, but instead a mature, logical approach to reading scientific works. At the end of a dozen pages the student has already read a passage from a standard scientific source. All emphasis is on how to *read* French and German. As Professor Locke says, "This approximation (to French pronunciation) often will not give a pronunciation understandable to a Frenchman, but may be sufficient for the communication of French words between Americans." It is this reviewer's sincere belief that these volumes fill a long-felt need and his hope that a similar volume will be forthcoming for the Russian language.

Computing with Desk Calculators. By Walter W. Varner. Rinehart, New York, 1957. $2.00.

"This manual is designed to furnish the user of the modern desk calculator

with sufficient understanding of the machine so that he can quickly design his own computing techniques. It is written primarily for a person who uses a machine to perform many different calculations instead of many repetitions of the same prototype."

Large-scale electronic computers are becoming more and more important, and users must become familiar with the desk calculator, which is customarily used for checking purposes. Particularly noteworthy is the inclusion of double precision calculations for numbers larger than the capacity of the desk calculator.

Canon Arithmeticus. By C. G. J. Jacobi. Akademie-Verlag, Berlin, 1956. 432 pp. D.M. 46.

These tables of indices for the prime moduli under 1000 should prove a welcome addition to any mathematical library.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

THE ACADEMY-RESEARCH COUNCIL

The National Academy of Sciences—National Research Council is a non-governmental, self-perpetuating organization of scientists, chartered by the U. S. Congress to further the natural sciences and to provide scientific and technical advice to the federal government.

One of the eight divisions of the Academy-Research Council is the Division of Mathematics, which has ten major scientific societies represented in its membership.

The Division has unique opportunities to stimulate interchange among sciences, and the activities of its various committees range from a campaign to acquaint young people with mathematics as a career to the preparation of a handbook of mathematical tables. The Division also serves in an advisory capacity to governmental agencies. At the present time, the Division is actively engaged in a campaign to insure a full representation of American mathematicians at the next International Mathematical Congress scheduled for Edinburgh, Scotland, in August 1958.

The Chairman of the Division is Professor P. A. Smith, Columbia University. The Executive Secretary of the Division is Professor H. W. Kuhn, Bryn Mawr College. The Division has appointed a Committee on Educational Policy to consider questions of training especially in secondary schools, and national education policy in mathematics. The Chairman of this Committee is Professor S. S. Cairns, University of Illinois.

PERSONAL ITEMS

Professor A. W. McGaughey, Bradley University, represented the Association at the inauguration of President Robert G. Bone of Illinois State Normal University on October 4, 1957.

The Catholic University of America: A program of late afternoon and evening courses leading to the master and doctoral degrees in mathematics will be offered in the fields of analysis, algebra, theory of probability, mathematical statistics, and numerical analysis. The program of public lectures in the field of mathematical statistics which were given monthly during the past year under the sponsorship of the National Science Foundation was resumed in September.

Michigan State University: Dr. Joseph Lehner, Staff Member, Los Alamos Scientific Laboratory, New Mexico, has been appointed Visiting Professor; Dr. Günther Ewald, University of Mainz, Dr. D. W. Hall, Amherst College, Dr. L. L. Helms, Senior Research Engineer, Convair Corporation, Pomona, California, and Dr. R. H. Wasserman, University of Michigan, have been appointed Assistant Professors; Mr. Yousef Alavi, Mrs. Delia W. Koo and Mr. R. E. Sechler have been appointed Instructors; Dr. Heinrich Larcher has been promoted to Assistant Professor.

University of Denver: The 4th Annual Symposium on Computers and Data Processing, sponsored by the Electronics Division, Denver Research Institute of the University, was held on August 29–30, 1957. This was followed by a special two weeks course on computer applications.

University of Michigan: Associate Professors R. C. F. Bartels, Raoul Bott, and Gail S. Young have been promoted to Professors; Assistant Professors J. W. Carr, III, A. J. Lohwater, and J. G. Wendel have been promoted to Associate Professors; Dr. J. W. Addison, Dr. A. B. Clarke, and Dr. E. L. Griffin, Jr., have been promoted to Assistant Professors; Visiting Assistant Professor D. G. Higman has been promoted to Assistant Professor; Dr. P. E. Conner, Institute for Advanced Study, has been appointed Assistant Professor; Dr. D. S. Greenstein, Professional Engineer, Radio Corporation of America, Camden, New Jersey, Dr. N. J. Hicks, Massachusetts Institute of Technology, and Dr. E. T. Parker, Graduate Student, Ohio State University, have been appointed Instructors; Assistant Professor F. M. Wright, Iowa State College, has been appointed Visiting Assistant Professor; Assistant Professor C. J. Coe has retired with the title Assistant Professor Emeritus; Associate Professor C. L. Dolph and Professor Erich Rothe have been granted Guggenheim Fellowships and are on sabbatical leave during 1957–58; Associate Professor W. J. LeVeque has received a Sloan Foundation Fellowship and is on leave of absence for 1957–58; Professor E. D. Rainville will be on sabbatical leave for the second semester of 1957–58; Dr. E. L. Griffin is on leave as Visiting Professor at Columbia University during 1957–58.

University of Minnesota: Dr. Steven Orey has been promoted to Assistant Professor; Associate Professors B. R. Gelbaum and G. K. Kalisch have been promoted to Professors; Mr. R. C. Bzoch, Illinois Institute of Technology, Dr. Jesus Gil de Lamadrid, Ohio State University, and Assistant Professor D. A. Storvick, Iowa State College, have been appointed Assistant Professors; Associate Professor G. F. Clanton, Baylor University, has been appointed Lecturer; Professor R. W. Brink, Chairman of the Department of Mathematics, has retired; Professor R. H. Cameron has been appointed Chairman of the Department; Professor J. M. H. Olmsted has been appointed Associate Chairman of the Department.

Associate Professor J. C. Abbott, U. S. Naval Academy, has been promoted to Professor.

Mr. D. S. Adorno, Iowa State College, has a position as a senior engineer at Sylvania Electric Products, Waltham, Massachusetts.

Mr. J. T. Ahlin, Applied Science Representative, I.B.M. Corporation, Houston,

Texas, is now General Manager of Applied Programming, I.B.M. Corporation, New York, New York.

Mr. Eugene Albert, Mathematician, General Electric Company, Schenectady, New York, has been appointed Assistant Professor at Union College.

Assistant Professor H. B. Anderson, Michigan College of Mining and Technology, has been promoted to Associate Professor.

Dr. P. H. Anderson, Survey Statistician, Department of the Army, Washington, D. C., is now employed as an analytical statistician, Bureau of Ships, Navy Department, Washington, D. C.

Dr. J. J. Andrews, Teaching Assistant, University of Georgia, is now employed by the Union Carbide and Carbon Corporation, Oak Ridge, Tennessee.

Mr. D. L. Arenson, Manager, Ex-Cel Development Company, Chicago, Illinois, has been appointed Assistant Manager at the American Machine and Foundry Company, Chicago.

Associate Professor J. W. Ault, U. S. Air Force Academy, has been promoted to Professor and Head of the Department of Mathematics.

Assistant Professor J. B. Bartoo, Pennsylvania State University, has been promoted to Associate Professor.

Associate Professor Lulu Bechtolsheim, University of Redlands, has been promoted to Professor.

Mr. J. S. Becker, Engineering Assistant, Studebaker Corporation, South Bend, Indiana, has been appointed Senior Engineer, Glenn L. Martin Company, Denver, Colorado.

Mr. J. G. Bennett, Computer, North American Aviation, Columbus, Ohio, has a position as a mathematician for the I.B.M. Corporation, Poughkeepsie, New York.

Assistant Professor S. K. Berberian, Michigan State University, has been appointed Assistant Professor, State University of Iowa.

Mr. C. J. Biggerstaff, Electronics Engineer, Rheem Manufacturing Company, Downey, California, is a magnetics chief project engineer for California Magnetics Control, North Hollywood, California.

Associate Professor A. H. Black, Southern Illinois University, has been promoted to Professor.

Dr. R. L. Blair, Michigan State University, has been appointed Assistant Professor at the University of Oregon.

Assistant Professor A. A. Blank, University of Tennessee, is on leave for one year as Research Scientist for the Institute of Mathematical Sciences, New York, New York.

Mr. A. F. Bond, Graduate Assistant, West Virginia University, is now employed as a mathematician for the Bendix Aviation Corporation, Ann Arbor, Michigan.

Dr. R. D. Boswell, Jr., Graduate Assistant, University of Georgia, has been appointed Associate Professor at Mississippi State College.

Dr. L. E. Boyer, Chairman of the Department of Mathematics, State Teachers College, Millersville, Pennsylvania, is now Advisor, College Preparatory, Department of Public Instruction, Harrisburg, Pennsylvania.

Rev. E. W. Brande, Graduate Student, St. Louis University, has been appointed Instructor at Canisius High School, Buffalo, New York.

Mr. J. P. Brannen, Sam Houston State College, has been promoted to Assistant Professor.

Mr. J. R. Brashear, Student, Carnegie Institute of Technology, is Director, Computation Department, R. A. Cummings, Jr. & Associates, Pittsburgh, Pennsylvania.

Mr. D. A. Breault, Carnegie Institute of Technology, has accepted a position as an engineer in the Missile Systems Laboratory, Sylvania Electric Products Company, Waltham, Massachusetts.

Dr. D. M. Brown, Research Engineer, Data Reduction and Computation, Willow Run Research Center, University of Michigan, Ypsilanti, Michigan, is now Educational Staff Consultant, Remington Rand Univac, St. Paul, Minnesota.

Mr. G. A. Brown, Graduate Student, Rutgers University, has a position as a mathematician at Okonite Company, Passaic, New Jersey.

Assistant Professor J. W. Brown, Clemson College, has been promoted to Associate Professor.

Mrs. Mary H. Brown, Perkinston Junior College, has been appointed Head of the Department of Mathematics.

Associate Professor H. D. Brunk, University of Missouri, has been promoted to Professor.

Associate Professor C. E. Burgess, University of Utah, has returned after his leave of absence as Visiting Lecturer at the University of Wisconsin.

Dr. J. R. Byrne, San Jose State College, has been appointed Assistant Professor at Portland State College.

Assistant Professor Lonnie Cross, Tuskegee Institute, has been appointed Associate Professor at Atlanta University.

Assistant Professor C. C. Faith, Michigan State University, has been appointed Assistant Professor at Pennsylvania State University.

Professor Ky Fan, University of Notre Dame, is on leave of absence and has accepted a position on the Mathematics Panel, Oak Ridge National Laboratory, Oak Ridge, Tennessee.

Assistant Professor J. R. Foote, University of Oklahoma, has accepted a position as Associate Professor in the Division of Engineering Sciences, Purdue University.

Dr. J. W. Gaddum, Michigan State University, has been appointed Assistant Professor at the University of Florida.

Dr. W. J. Harrington, Pennsylvania State University, has been appointed Professor at North Carolina State College.

Assistant Professor L. H. Kanter, Drexel Institute of Technology, has been promoted to Associate Professor.

Professor Don Kirkham, Iowa State College, has received a Guggenheim Fellowship for research and a Fulbright award for lecturing in soil physics at the State Agricultural University, Ghent, Belgium, for the academic year 1957–58.

Dr. Paul Koosis, Research Assistant, Institute of Mathematical Sciences, New York University, has received a Fulbright grant and will conduct research at the University of Montpellier, France, during the academic year 1957–58.

Mr. Arthur Libenson, Bedford, Massachusetts, has accepted a position as systems engineer at Raytheon Manufacturing Company, Maynard, Massachusetts.

Mr. R. T. J. Mahoney, Teaching Fellow, University of Buffalo, has been appointed Assistant Professor at the U. S. Naval Academy.

Assistant Professor J. H. McKay, Michigan State University, has been appointed Assistant Professor at Seattle University.

Dr. J. A. Schatz, University of Connecticut, has a position as a staff member at the Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor Edward Silverman, Michigan State University, has been appointed Associate Professor at Purdue University.

Professor F. C. Smith, College of St. Thomas, has a position as Actuary at George V. Stennes & Associates, Minneapolis, Minnesota.

Mr. L. T. Wos, University of Illinois, has accepted a position as Assistant Mathematician at the Argonne National Laboratories, Lemont, Illinois.

Dean Elijah Swift, University of Vermont, died on July 21, 1957. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-EIGHTH SUMMER MEETING OF THE ASSOCIATION

The thirty-eighth summer meeting of the Mathematical Association of America was held at Pennsylvania State University, University Park, Pennsylvania, from Monday, August 26 through Thursday, August 29, 1957, in conjunction with the summer meetings of the American Mathematical Society, the Society for Industrial and Applied Mathematics, and the Pi Mu Epsilon Fraternity. There were registered 924 persons, including 451 members of the Association.

Sessions of the Association were held on Monday morning and afternoon, on Tuesday morning, and on Wednesday and Thursday afternoons. All sessions were held in 121 Sparks Building of Pennsylvania State University. Presiding officers were President G. B. Price, Vice-President B. W. Jones, and Professors Deane Montgomery and W. L. Duren. The sixth series of Earle Raymond Hedrick Lectures was delivered by Professor Leo Zippin of Queens College. The lecture by Professor Albert Edrei was presented on closed circuit television. The sessions on Wednesday and Thursday afternoons were open meetings of the Association's Committee on the Undergraduate Program in Mathematics. The Program Committee for the meeting consisted of N. J. Fine, Chairman; Harley Flanders, and W. R. Transue.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: *Topological transformation groups;* Lecture I, by Professor Leo Zippin, Queens College.

Elementary problems in the theory of entire functions, by Professor Albert Edrei, Syracuse University.

Intersection properties of convex sets, by Professor V. L. Klee, Jr., University of Washington.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Zippin.

The science teaching improvement program, by Dr. J. R. Mayor, Director, Science Teaching Improvement Program, A.A.A.S.

A special program for gifted undergraduates, by Professor K. O. May, Carleton College.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Zippin.

Business Meeting of the Association.

Survey of research potential and training in the mathematical sciences, by Professor J. W. Green, University of California, Los Angeles.

Mathematicians in industry and government, by Dean Mina S. Rees, Hunter College. (By title).

FOURTH SESSION OF THE ASSOCIATION

Teaching statistical inference in elementary mathematics courses, by Professor S. S. Wilks, Princeton University.

Symposium: *Probability and statistics in undergraduate mathematics.*
Introduction by Professor M. E. Munroe, University of Illinois.
Moderator: Dr. Brockway McMillan, Bell Telephone Laboratories.

Symposium: *The study of logic and foundations in undergraduate courses.*
Introduction by Professor P. R. Halmos, University of Chicago.
Moderator: Professor J. G. Kemeny, Dartmouth College.

FIFTH SESSION OF THE ASSOCIATION

The work of the committee on the undergraduate program in mathematics: C.U.P. books and commercial texts, by Professor R. L. Davis, University of Virginia.

The C.U.P. program, by Professor J. G. Kemeny, Dartmouth College.

Symposium: *Goals of freshman courses.*
Introduction by Professor K. O. May, Carleton College.
Moderator: Professor G. B. Price, University of Kansas.

Symposium: *Calculus for freshmen.*
Introduction by Professor R. V. Andree, University of Oklahoma.
Moderator: Professor W. L. Duren, Jr., University of Virginia.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Beta Theta Pi fraternity house with twenty-six members present. Among the more important items of business transacted were the following:

To fill a vacancy, the Board elected Professor Ernst Snapper of Miami University as Governor from the Ohio Section for a one-year term.

It was voted to authorize future meetings of the Association at the Massachusetts Institute of Technology in August 1958, at the University of Pennsylvania in January 1959, and at the University of Utah in August 1959.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Tuesday morning, with President G. B. Price presiding. The Secretary reported that the membership of the Association was 7,024 on August 23, 1957.

The following committee chairmen reported for their committees: Professor B. W. Jones, Committee on Visiting Lecturers; Professor J. S. Frame, Committee on Employment Opportunities; Professor Edith R. Schneckenburger, Committee on Sections.

At the conclusion of the Tuesday morning session, a protracted discussion was held of some of the problems of mathematicians in industry and government. The following motion presented by Professor Wallace Givens was adopted: "It is the sense of this meeting that the Albert Survey has proved of sufficient value that a continuing or recurrent analysis of a similar sort is fully justified. It is particularly important that any such further study pay special attention to the employment of mathematicians in industry and other non-academic positions."

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the Hetzel Union Building. Fifty-five persons were present representing 26 of the 27 Sections.

The national contest for high school students planned for the spring of 1958 was discussed in great detail. The visiting lecturer programs for high schools sponsored by the Kentucky Section and the Northern California Section were described. Several sectional committees reported on their activities pertaining to the strengthening of the teaching of mathematics.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. The colloquium lectures on *Cohomology operations* were delivered by Professor N. E. Steenrod of Princeton University. Invited addresses were given by Professors A. P. Calderon and M. A. Rosenlicht.

The Society for Industrial and Applied Mathematics and the Pi Mu Epsilon Fraternity met on Monday and Tuesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Evan Johnson, Jr., Chairman; J. B. Bartoo, Orrin Frink, H. M. Gehman, W. O. Gordon, H. L. Krall, R. D. Schafer.

Registration headquarters were located in Waring Hall, with dormitory and cafeteria accommodations in the same area. The Employment Register and the Book Exhibit were on display in Waring Hall.

On Monday evening, Sigma Delta Epsilon Honorary Society held a tea in Hamilton Hall for the women mathematicians. A reception for members of the mathematical organizations was held in the Hetzel Union Building on Wednesday afternoon immediately preceding the dinner. On Thursday afternoon a chicken barbecue was held in Horticultural Woods.

At the banquet on Wednesday evening, Professor H. B. Curry acted as toastmaster. The visitors were welcomed on behalf of the University by Dean Ben Euwema, Professor Orrin Frink, and President Eric A. Walker. Professors E. J. McShane responded for the American Mathematical Society, G. B. Price for the Mathematical Association of America, T. H. Southard for the Society for Industrial and Applied Mathematics, and J. S. Frame for the Pi Mu Epsilon Fraternity.

A resolution prepared by Professor W. L. Hart was unanimously adopted expressing sincere appreciation of the hospitality shown by Pennsylvania State University, and presenting cordial thanks to Professor Evan Johnson and his Committee on Arrangements for their efforts, which commenced long in advance and ensured the success of the meetings.

HARRY M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 52 persons have been elected to membership by the Board of Governors on applications duly certified.

RICHARD P. AMENT, A.B. (Cornell U.) Instr., College of Wooster.

JOHN P. ANDERSEN, Student, University of Omaha.

ALLENE B. ARCHER, M.Ed. (Virginia) Professor, Maryland State Teachers College, Towson.

DAN B. AYDELOTT, B.S. (Ozarks) Nuclear Engr., Convair, Ft. Worth, Texas.

ROBERT C. BALENTINE, Student, Baylor University.

DONALD L. BARNETT, Student, University of Missouri.

STANLEY D. BATTLE, Student, Baylor University.

HOMER F. BECHTELL, JR., M.S. (Wisconsin) Teaching Asst., University of Wisconsin.

JEROME G. BEERY, Student, Kansas State College.

WILLIAM M. BOLING, Student, Baylor University.

CHARLES M. BRADEN, M.S. (Minnesota) Asst. Professor, Macalester College.

FREDERICK W. BRUNDAGE, B.A. (Alfred) Grad. Student, University of Rochester.

CHARLES W. BURMEISTER, Student, Baylor University.

JAMES T. BYRNE, Student, Kent State University.

KENNETH C. CARTWRIGHT, B.S.E.E. (Purdue) Vandervoort, Arkansas.

LEONARD A. CASCIOTTI, Student, Pennsylvania State University.

MRS. ANGELA L. CHANG, B.A. (Alfred) Jr. Math., Cornell Aeronautical Lab., Buffalo, N. Y.

JAMES D. CHURCH, B.A. (Nebraska) University of Nebraska.

JOSEPHINE A. CURRAN, A.B. (George Washington) Chevy Chase, Maryland.

MAHLON M. DAY, PH.D. (Brown) Professor, University of Illinois.

JOHN W. DEFORD, Student, Carleton College.

SUDHISH G. GHURYE, PH.D. (North Carolina) Asst. Professor, University of Chicago.

ALLEN A. GOLDSTEIN, PH.D. (Georgetown) Design Specialist, Convair Astronautics, San Diego, Calif.

WILLIAM B. GRAGG, JR., Student, University of Denver.

ELI HELLERMAN, B.S. (George Washington) Math.-Programmer, Council for Economic and Industry Research, Arlington, Va.

THOMAS J. HILL, B.S., M.ED. (Oklahoma) Teacher, Classen High School, Oklahoma City, Okla.

EARL D. HILLER, M.S. (Oklahoma) Instr., Oklahoma City University.

JAMES T. HODDE, Student, Baylor University.

DONALD G. HOOK, B.S. (Huron) Grad. Asst., South Dakota State College.

GEORGE W. HORTON, JR., M.S. (Oklahoma A. & M.C.) Asst. Professor, Upper Iowa University.

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THE EIGHTEENTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The eighteenth annual William Lowell Putnam Mathematical Competition will be held on Saturday, February 8, 1958.* This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed out early in December to the regular mailing list. If an application blank is not received by December 15, you may secure one from the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio by a postcard request. Your application must be filed with the director not later than January 15, 1958. For further details of the examination and the list of prizes (including the $2500 scholarship at Harvard) see the announcement which will be mailed out along with the application blank.

Reports of the seventeen previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942, October 1946, August–September 1947, December 1948, August–September 1949, 1950, 1951, October 1952, 1953, 1954, 1955, December 1956, and August–September (announcement of winners) and November (questions and solutions) 1957.

THE NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section, Mathematical Association of America, was held at Muhlenberg College on November 24, 1956 with Professor John Oxtoby, Bryn Mawr College, Chairman of the Section, presiding. There were 64 present including 50 members of the Association.

The following slate of officers was elected: Chairman, Professor Albert Wilansky, Lehigh University; Members Executive Committee, Professor C. W. Saalfrank, Lafayette College (two years to complete the unexpired term of N. J. Fine), Professor E. R. Mullins, Jr., Swarthmore College (three years). Revisions of the By-Laws were approved.

The matter of Section participation in the proposed National Contest for High Schools was discussed and the following resolution adopted:

The Philadelphia Section of the Mathematical Association of America agrees in principle with the establishment of a National Contest for High Schools. The Section believes that considerable autonomy should be left with the Sections to determine the particular form of organization most suitable for the Section within the national regulations.

Professor G. C. Webber discussed briefly the examination for high school students given in Delaware in the spring of 1956 under partial sponsorship of the Mathematical Association of America.

The following papers were presented at this meeting:

1. *Integers*, by Professor Peter Scherk, Universities of Saskatchewan and Pennsylvania, introduced by the Secretary.

The author discussed sets of integers, their sums and counting functions. Some early results by Landau and Besicovitch were reviewed and more recent results and problems connected with Mann's theorem were surveyed (cf. *Ann. Math.*, vol. 43, 1942, pp. 523–527).

2. *How to tell that a simple overhand knot is really knotted*, by Professor E. E. Moise, University of Michigan and Institute for Advanced Study.

This was an expository talk, giving a proof that a trefoil knot does not have the knot-type

* In order to get out the results of the Putnam Competition in time for it to be of maximum use, it is probable that the nineteenth competition will be held about December 1, 1958.

of the circle. The method was to show that the fundamental group of the complement of the trefoil is not commutative.

3. *On the Cauchy criterion for the convergence of an infinite series*, by Professor Albert Wilansky, Lehigh University.

Proofs were given of a theorem of S. Mazur and W. Orlicz, *Studia Mathematica*, vol. 14, 1954, Theorem 4.3 p. 158, to the effect that convergence of a sequence x cannot be concluded from a countable set of restrictions on the sequence of the form "$x_{p_n}-x_{q_n}\to 0$, p_n, q_n being increasing sequences of indices"; or, more generally, of the form "Ax is a null sequence, x being a column vector and A a matrix which transforms each convergent sequence into a null sequence." The general principle of convergence supplies a set of such conditions with the power of the continuum.

4. *Impossibility of computational algorithms for group-theoretic problems*, by Dr. M. O. Rabin, Princeton University, introduced by the Secretary.

G. C. WEBBER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-first Annual Meeting, University of Cincinnati and Hotel Sheraton-Gibson, Cincinnati, Ohio, January 31, 1958.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Washington and Jefferson College, Washington, Pennsylvania, May, 1958.

ILLINOIS, Illinois College, Jacksonville, May 9–10, 1958.

INDIANA, May 10, 1958.

IOWA, Drake University, Des Moines, April 18, 1958.

KANSAS

KENTUCKY, University of Kentucky, Lexington, April, 1958.

LOUISIANA-MISSISSIPPI, Loyola University, New Orleans, February 21–22, 1958.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Georgetown University, Washington, D. C., December 7, 1957.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March, 1958.

MINNESOTA

MISSOURI, University of Missouri, Columbia, Spring, 1958.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1958.

NEW JERSEY, Fairleigh Dickinson University, Rutherford, November 2, 1957.

NORTHEASTERN, Dartmouth College, Hanover, New Hampshire, November 30, 1957.

NORTHERN CALIFORNIA, San Francisco State College, January 18, 1958.

OHIO, Denison University, Granville, April, 1958.

OKLAHOMA

PACIFIC NORTHWEST, Oregon State College, Corvallis, June 20, 1958.

PHILADELPHIA, November 30, 1957, Harvard College, Haverford, Pennsylvania.

ROCKY MOUNTAIN, Colorado State College, Greeley, Spring, 1958.

SOUTHEASTERN, University of Florida, Gainesville, March 14–15, 1958.

SOUTHERN CALIFORNIA, Pasadena City College, March 8, 1958.

SOUTHWESTERN, University of New Mexico, Albuquerque, April 11–12, 1958.

TEXAS, Baylor University, Waco, April, 1958.

UPPER NEW YORK STATE, Ecole Polytechnique and University of Montreal, Montreal, Quebec, Canada, May, 1958.

WISCONSIN, Carroll College, Waukesha, May, 1958.

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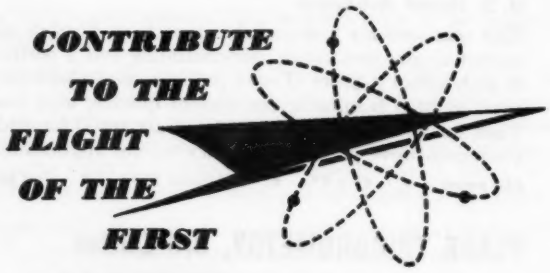

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